

## SEMI-CALABI-YAU VARIETIES AND MIRROR PAIRS

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ABSTRACT. We prove cohomological mirror duality for varieties of Borcea–Voisin type in any dimension. Our proof applies to all examples which can be constructed through Berglund–Hübsch duality. Our method is a variant of the Landau–Ginzburg model and of the Landau–Ginzburg/Calabi–Yau correspondence which allows us to prove the classical cohomological mirror symmetry statement for an orbifold version of the ramification locus of the anti-symplectic involution. These ramification loci mirroring each other are beyond the Calabi–Yau category and feature sextic curves in  $\mathbb{P}^2$ , octic surfaces in  $\mathbb{P}^3$ , degree-10 three-folds in  $\mathbb{P}^4$ , etc.

## 1. INTRODUCTION

Mirror pairs  $(X, X')$  of smooth projective  $D$ -dimensional varieties satisfying

$$(1) \quad h^{p,q}(X; \mathbb{C}) = h^{D-p,q}(X'; \mathbb{C})$$

constitute some of the most influential and inspirational mathematical phenomena deriving from physics. Since the origin of mirror symmetry, in the early 1990s, Calabi–Yau varieties or orbifolds played a special role: a wide range of constructions of pairs  $X$  and  $X'$  of Calabi–Yau type were put forward by both physicists and mathematicians and the equation (1) was established at least in terms of Chen–Ruan orbifold cohomology. (This implies the identity in ordinary cohomology for crepant resolutions whenever they exist.) In [22], Givental extended mirror symmetry from Calabi–Yau varieties to Fano varieties; there, however, the duality becomes asymmetric: a Fano  $X$  is mirror dual to an object of the form

$$W : U \rightarrow \mathbb{C},$$

a regular complex-valued function on a quasi-projective variety usually referred to as the *Landau–Ginzburg* model. The main invariant attached to it is the Jacobi ring. It is sometimes possible to go back from a Landau–Ginzburg model to the cohomology of an ordinary variety (or orbifold). For instance, in [23], Gross, Katzarkov and Ruddat describe mirror symmetry for some cases of general type varieties  $X$ : there, the mirror  $X'$  needs to be equipped with a sheaf of vanishing cycles and the main invariant is a Hodge filtration of hypercohomology groups.

In this paper we show that the same classical mirror symmetry duality (1) applies in a range of cases lying beyond the category of Calabi–Yau varieties. The construction is strikingly simple and it is formulated here by adapting Berglund–Hübsch duality [5]. The prototype example are smooth hypersurfaces  $X$  of degree  $d$  within  $\mathbb{P}^{2d-1}$ ; their analogues are degree- $d$  hypersurfaces in  $\mathbb{P}(w_1, \dots, w_n)$  where the weighted number  $\sum_j w_j$  of homogeneous coordinates is one half of the degree  $d$ . We will refer to them as  $1/2$ -Calabi–Yau and they will be shown to admit a mirror partner  $X'$  of the same kind satisfying the original classical mirror symmetry duality (1). In general  $X$  and  $X'$  may be disconnected but possess a distinguished component of dimension  $D$ .

The earliest examples of these  $1/2$ -Calabi–Yau mirror pairs are actually well known and widely studied curves appearing in the context of mirror symmetry of K3 surfaces with lattice polarisation. Let  $S$  and  $S'$  be two K3 surfaces with anti-symplectic involutions  $\sigma$  and  $\sigma'$ ; we assume that  $(\Sigma, \sigma)$  and  $(\Sigma, \sigma')$  mirror each other in the sense of mirror symmetry with lattice polarisation (for which we provide a short discussion in the appendix). The  $\sigma$ -fixed locus  $X$  and the  $\sigma'$ -fixed locus  $X'$  are two disjoint unions of curves whose number of components, *i.e.*  $h^{0,0}$ , and whose total genus, *i.e.*  $h^{1,0}$ , are interchanged; in other words  $X$  and  $X'$  satisfy (1) for  $D = 1$ .

In the literature, much attention was devoted to a special case of this phenomenon but from a slightly different point of view: Borcea–Voisin mirror symmetry. There, out of  $(\Sigma, \sigma)$  and  $(\Sigma', \sigma')$ , and for any elliptic curve  $E$  with its hyperelliptic involution  $\iota$ , we can produce some of the earliest mirror pairs of Calabi–Yau three-folds satisfying (1): the crepant resolutions of  $(\Sigma \times E)/(\sigma \times \iota)$  and  $(\Sigma' \times E)/(\sigma' \times \iota')$ . These Borcea–Voisin mirror pairs were systematically studied. In some cases, these mirror three-folds, after resolution, feature among better known constructions of mirror hypersurfaces. In a wide range of cases they can be treated systematically in toric geometry (see Borcea [10]). The proof of the statement follows from Nikulin’s classification, lattice mirror symmetry for K3 surfaces, Dolgachev [19] and Voisin [36].

Berglund–Hübsch duality, however, produces a number of examples of explicit and potentially different mirror symmetry statements. Their study could only be treated by explicit case-by-case analysis. This happens because the crepant resolution  $X$  and  $X'$  are often embedded into non-Gorenstein ambient spaces, which prevents us from applying straight away Borisov–Batyrev [4] polar duality for reflexive polytopes.

Then, the only viable approach appeared to be the explicit resolutions of the expressions appearing in Nikulin’s explicit list of the 79 surfaces of type K3 with anti-symplectic involutions. This programme was recently carried out completely in a series of papers by Artebani, Boissière and Sarti [1, 2]. This is a difficult and often spectacular check of mirror symmetry, but it is hard to imagine how a thorough treatment could be provided without discussing only the most representative examples and without referring to several computer-based computations. Most importantly, the lack of a conceptual, unified approach prevents us from generalising the statement to higher-dimensions and other group quotients (indeed, Comarin, Lyons, Priddis and Suggs made progress based on explicit resolutions in the direction of other cyclic non-symplectic actions [15]).

It is interesting to notice that Borcea himself started straight away testing higher dimensional mirror pairs of the form  $[(\Sigma \times T)/(\sigma \times \vartheta)]$  and  $[(\Sigma' \times T')/(\sigma' \times \vartheta')]$  with  $\Sigma$  and  $T$  Calabi–Yau of any dimensions, [10, §10]. Berglund–Hübsch construction, which also appeared in the 1990s, provided at that time many open examples of this kind to compute. A few have been tested explicitly, see for instance [31, 9, 10, 16]. The resolutions of the quotients  $(\Sigma \times T)/(\sigma \times \vartheta)$  have been systematically studied in the algebraic setup by Cynk and Hulek [17]. In 2013, Camere [12] has defined lattice polarised mirror symmetry in higher dimension with explicit results in the case of for  $\text{Hilb}^2(\Sigma)$  for  $\Sigma$  of K3 type (see also Boissière, Camere, and Sarti [8]).

We prove the mirror duality (1) between  $[(\Sigma \times T)/(\sigma \times \vartheta)]$  and  $[(\Sigma' \times T')/(\sigma' \times \vartheta')]$  in any dimension. To this effect we use mirror pairs of  $1/2$ -Calabi–Yau  $(X, X')$  and  $(Y, Y')$ , where  $X$  and  $Y$  occur essentially as the branch locus within two weighted projective spaces  $\mathbb{P}(\mathbf{v})$  and  $\mathbb{P}(\mathbf{w})$  covered by  $\Sigma \rightarrow \mathbb{P}(\mathbf{v})$  and  $T \rightarrow \mathbb{P}(\mathbf{w})$ . The precise setup and definition of  $X$  is given in §3, but the prototype of these  $1/2$ -Calabi–Yau orbifolds are the sextic curve  $X^6 \subset \mathbb{P}^2$  or the octic surface  $X^8 \subset \mathbb{P}^3$  where a two-sheeted K3 covering of  $\mathbb{P}^2$  and a two-sheeted Calabi–Yau

covering of  $\mathbb{P}^3$  are respectively branched. In Theorem 17, we get

$$(2) \quad H_{\text{CR}}^{p,q}(X; \mathbb{C}) \cong H_{\text{CR}}^{D-p,q}(X'; \mathbb{C}),$$

where  $D$  is the maximum of the dimensions of the connected components of  $X$  and  $X'$ . Further to (2) we prove some related correspondences of mirror type involving the anti-symplectic involution. As a corollary we get the mirror statement relating the two Calabi–Yau orbifolds  $[\Sigma \times T/(\sigma \times \vartheta)]$  and  $[\Sigma' \times T'/(\sigma' \times \vartheta')]$ . This statement is extremely elementary and can be reproduced here below in full.

**Setup.** For  $n > 2$ , write  $\Sigma$  for an orbifold given by modding out a smooth Calabi–Yau hypersurface  $V(f)$  within  $\mathbb{P}(v_0, v_1, \dots, v_n)$  by a finite abelian group  $G$ . The hypersurface is defined by a polynomial of the form  $f = x_0^2 + \sum_{i=1}^n \prod_{j=1}^n x_j^{e_{i,j}}$  quasi-homogenous for a unique choice of weights  $v_0, v_1, \dots, v_n$ . The group  $G$  is formed by diagonal symmetries of  $f$  lying in  $\text{SL}(\mathbb{C}; n+1)$  and containing the monodromy operator attached to  $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  (a fibration off the origin). Notice that  $\sigma: \Sigma \rightarrow \Sigma$  given by changing the sign of  $x_0$  and fixing the remaining co-ordinates is an anti-symplectic involution. We get in this way an action  $a_G: G \rightarrow \text{Aut } V(v)$  and the orbifold  $\Sigma$  is precisely defined as the quotient stack  $[V(v)/\overline{G}]$  where  $\overline{G} = G/\ker a_G$ .

The data for  $T$  is completely analogous, and denoted using:  $\{g = 0\} \subset \mathbb{P}(w_1, \dots, w_m)$  with  $H$ -action and faithful  $\overline{H}$ -action leading to  $T = [\{g = 0\}/\overline{H}]$ .

**Mirrors.** The mirror of  $\Sigma$  is a Calabi–Yau  $\Sigma'$  of the same form as  $\Sigma'$ : a hypersurface modulo a group of diagonal symmetries. The hypersurface is defined by a transposition of the exponents: namely, the vanishing locus of  $f' = x_0^2 + \sum_{i=1}^n x_j^{e_{j,i}}$  within  $\mathbb{P}(v'_0 + v'_1, \dots, v'_n)$ . The group  $G'$  is defined by the standard Cartier duality:  $G$  is injected via  $i_G$  into the finite group of all diagonal symmetries of  $f$  and  $G'$  is the kernel of the Cartier dual of  $i_G$ . For  $n > 2$ ,  $\Sigma'$  is Calabi–Yau because  $\Sigma$  is.

We can repeat the same construction for  $T$ ; we get  $\{g' = 0\} \subset \mathbb{P}(w'_1, \dots, w'_m)$  with  $H'$ -action and  $T' = [\{g' = 0\}/\overline{H}']$ . The following theorem is a consequence of Corollary 22.

**Theorem.** *The Chen–Ruan orbifold cohomology of the  $(n + m - 2)$ -dimensional quotient stacks  $V = [(\Sigma \times T)/(\sigma \times \vartheta)]$  and  $V' = [(\Sigma' \times T')/(\sigma' \times \vartheta')]$  satisfy*

$$H_{\text{CR}}^{p,q}(V; \mathbb{C}) \cong H_{\text{CR}}^{n+m-2,q}(V'; \mathbb{C}).$$

Furthermore, for  $n, m > 2$ , whenever the coarse spaces of  $V$  and  $V'$  admit crepant resolutions we obtain two  $n + m - 2$  dimensional Calabi–Yau manifolds  $\tilde{V}$  and  $\tilde{V}'$  satisfying

$$H^{p,q}(\tilde{V}; \mathbb{C}) \cong H^{n+m-2,q}(\tilde{V}'; \mathbb{C}).$$

**Landau–Ginzburg models of  $1/2$ -Calabi–Yau.** Landau–Ginzburg (LG) models play the role of vehicle between the geometric mirror duals. We already mentioned them in the above discussion as mirror partners of Fano varieties. But in recent years they have been treated as independent objects deserving a fully developed mathematical theory analogous to cohomology and quantum cohomology of projective varieties (we refer to FJR theory [21]). This led to LG-to-LG mirror symmetry relating LG models with each other. In purely cohomological terms this statement was proven by Krawitz [27] and Borisov [11] and matches for instance the bi-graded  $G$ -orbifolded Jacobi ring  $\text{Jac}_G(f)$  in the above setup with  $\text{Jac}_{G'}(f')$  (see §5.2):  $(p, q)$ -classes are mapped to  $(n - 1 - p, q)$ -classes; we summarize the statement in Theorem 25 and we insist on its generality: no Calabi–Yau condition is involved.

Then, Chiodo–Ruan state correspondence is used to connect the LG model  $f: [\mathbb{C}^n/G] \rightarrow \mathbb{C}$  to  $[\{f=0\}/\overline{G}]$ . In this way we can translate LG-to-LG mirror symmetry backwards into mirror symmetry of projective varieties. Unfortunately, this requires a restrictive Calabi–Yau condition. This explains the special role of Calabi–Yau in mirror symmetry from the present point of view. As it was noted at the time of its proof, the state correspondence is slightly more general than physicists predicted:  $\{f=0\}$  is assumed to be a Calabi–Yau, but no Calabi–Yau condition such as  $G \subset \mathrm{SL}(\mathbb{C}; n)$  imposed on the symmetry group (see [14, Rem. 1]). This left some reasonable hope that more geometric consequences could be derived from LG-to-LG mirror symmetry.

In view of  $1/2$ -Calabi–Yau mirror symmetry, the crucial idea is to consider slight generalizations of the usual state spaces  $\mathrm{Jac}_G(f) := \bigoplus_{\gamma \in G} (\mathrm{Jac} f_\gamma)^G$ , where  $f_\gamma$  is the restriction to the state space. Namely, on the one hand we consider  $G$  extended by  $\sigma$ , exploiting the above observation that the symmetry group does not need to lie in  $\mathrm{SL}$ . On the other hand, we do not extract  $G[\sigma]$ -invariant forms  $(\mathrm{Jac} f_\gamma)^{G[\sigma]}$ , but rather include all  $G$  invariant forms. We write  $\mathcal{H}_S(f)^G$  for the new state space with  $G$  as above and  $S = G[\sigma]$ ; the new LG model (see Theorem 25) invariant has a mirror statement of the form

$$\mathcal{H}_S(f)^G \quad \text{mirrors} \quad \mathcal{H}_{G'}(f)^{S'},$$

with  $S'$  and  $G'$  defined via Cartier duality as above. The left hand side is proven to encode all the relevant cohomological information of a  $1/2$ -Calabi–Yau model  $X$  attached to  $f$  and  $G$ , while the right hand side (after applying Proposition 14 and Lemma 31) is proven to encode the analogous information for the mirror  $1/2$ -Calabi–Yau model  $X'$  attached to  $f'$  and  $G'$  (see §3). The mirror symmetry Theorem 17 is indeed stronger than the usual theorem for Calabi–Yau and we deduce the theorem stated above.

**Structure of the text.** Section 2 establishes the terminology (see in particular Chen–Ruan orbifold cohomology and slight variants of it). Section 3 defines the setup of  $1/2$ -Calabi–Yau models. Section 4 provides the  $1/2$ -Calabi–Yau mirror symmetry statement, the proof is deferred to the next section, but we deduce from this statement the ordinary Calabi–Yau mirror theorem stated above. Section 5 proves mirror symmetry for  $1/2$ -Calabi–Yau pairs using slight modifications of LG models. The appendix provides a brief presentation of mirror symmetry of lattice polarised K3 surfaces.

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## 2. TERMINOLOGY

**2.1. Conventions.** In this paper all schemes and algebraic spaces are assumed to be of finite type over the complex numbers  $\mathbb{C}$ . All algebraic groups are linear groups, *i.e.* they are isomorphic to closed subgroups of  $\mathrm{GL}(n; \mathbb{C})$  for some  $n$ . We systematically write  $\mathbb{G}_m$  for the multiplicative group  $\mathbb{C}^\times \subset \mathbb{C}$ . In fact, almost all algebraic groups used here are subgroups of a torus  $(\mathbb{G}_m)^n$  for some  $n$ .

**2.2. Finite order diagonal actions.** We often deal with diagonal matrices of finite order on the complex numbers. The entries  $\alpha_j$  along the diagonal are necessarily  $D$ th roots of unity  $\alpha_j \in \mu_D$  for some  $D$ ; we write

$$(3) \quad \frac{1}{D}(p_1, \dots, p_m) \in \mathrm{GL}(m; \mathbb{C}),$$

or simply  $(\frac{p_1}{D}, \dots, \frac{p_m}{D})$ , for the symmetry acting on the  $j$ th co-ordinate as  $\alpha_j = \exp(2\pi i p_j/D)$ . This is a good spot to define the age of the symmetry  $\frac{1}{D}(p_1, \dots, p_m)$ .

*Definition 1.* The age of  $\frac{1}{D}(p_1, \dots, p_m)$  is defined as

$$a(\frac{1}{D}(p_1, \dots, p_m)) = \sum_j \frac{p_j}{D} \text{ for } p_j \in \{0, \dots, D-1\}.$$

For any polynomial  $P$  in  $m$  variables and for any  $\gamma = \frac{1}{D}(p_1, \dots, p_m)$  acting on the domain of  $P$ , we denote by  $(P)_\gamma$  the restriction to the fixed space  $(\mathbb{C}^m)_\gamma$  spanned by the fixed variables  $x_j \mid \gamma^* x_j = x_j$ . We often use the set of labels of the fixed variables, and we denote it by

$$F_\gamma = \{j \mid \gamma^* x_j = x_j\}.$$

**2.3. The stacks we consider and their inertia stacks.** The geometric objects studied in this paper are Deligne–Mumford stacks of the form

$$\mathfrak{X} = [U/G],$$

where  $G$  is an abelian linear algebraic group as above,  $U$  is a smooth scheme and  $G$  acts properly on  $U$  (the map  $G \times U \rightarrow U \times U$ ,  $(g, x) \rightarrow (x, g \cdot x)$  is proper).

In general, the inertia group scheme of  $\mathfrak{X}$  is given by  $I_{\mathfrak{X}} := \mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$ . When we consider  $\mathfrak{X}$  of the form  $[U/G]$  we simply get the quotient stack

$$I_{\mathfrak{X}} = [I_U(G)/G], \text{ where } I_U(G) = \{(x, g) \mid g \cdot x = x\}.$$

Note that, since the  $G$ -action is proper, the projection  $I_U(G) \rightarrow U$  is finite.

In most cases we actually work in the following setup

$$\mathfrak{X} = [U/S\mathbb{G}_m]$$

where  $U$  is a smooth variety  $V(f) \subseteq \mathbb{C}^n \setminus \mathbf{0}$  for some weighted homogeneous polynomial  $f$  of degree  $d$  and of weights  $w_1, \dots, w_n$  in  $n$  variables,  $S$  is a finite group of diagonal actions preserving  $f$ , and, finally,  $\mathbb{G}_m$  acts on  $\mathbb{C}^n$  with weights  $w_1, \dots, w_n$ . Then, the inertia stack can be explicitly presented as

$$I_{[U/S\mathbb{G}_m]} = \bigsqcup_{\gamma \in S\mathbb{G}_m} [U/S\mathbb{G}_m]_\gamma = \bigsqcup_{\gamma \in S\mathbb{G}_m} [V(f_\gamma)/S\mathbb{G}_m],$$

where  $V(f_\gamma)$  is an affine hypersurface in  $(\mathbb{C}^n)_\gamma$  and should obviously be regarded as the empty set if  $\gamma$  fixes only the origin. The inertia group scheme  $I_{S\mathbb{G}_m}(U)$  mentioned above is

$$I_{S\mathbb{G}_m}(U) = \bigsqcup_{\gamma \in S\mathbb{G}_m} V(f_\gamma),$$

it is interesting in its own right (see for instance [20]), and it plays a role in the proof of Theorem 17.

**2.4. Orbifold cohomology and Chen–Ruan cohomology.** If we ignore the grading, Chen–Ruan cohomology of  $\mathfrak{X}$  is simply the cohomology of the inertia stack  $I_{\mathfrak{X}}$ . Notice that the only group elements  $\gamma$  fixing a nonempty set in  $U$  are finite order representations of the tangent space at the fixed point. Therefore, using the convention established above, we have  $\gamma = \frac{1}{D}(p_1, \dots, p_n)$  for some indices  $D, p_i$  and for some  $n$ . In all the above situations, the stack  $\mathfrak{X}$  is smooth and its tangent bundle pulls back to the inertia stack via the forgetful morphism  $p: I_{\mathfrak{X}} \rightarrow \mathfrak{X}$ . There, at each point, the element  $\gamma$  may be regarded as a finite-order representation of the  $(n-1)$ -dimensional fibre of  $T_{\mathfrak{X}}$  which can be written as  $\gamma = \frac{1}{D}(p_1, \dots, p_{n-1})$ . The age  $a(\gamma)$  of this representation is constant on each connected component and in this case depends only on  $\gamma$ . We now set Chen–Ruan orbifold cohomology, which is the cohomology of the inertia stack after a degree-shift,

$$H_{\text{CR}}^{p,q}([U/S\mathbb{G}_m]; \mathbb{C}) := \bigoplus_{\gamma \in H\mathbb{G}_m} H^{p-a(\gamma), q-a(\gamma)}(V(f_\gamma)/S\mathbb{G}_m; \mathbb{C}).$$

Notice also that each summand is the ordinary cohomology of a finite group quotient by  $H$  of the projective variety  $V(f_\gamma)/\mathbb{G}_m$  defined within the weighted projective subspace

$$\mathbb{P}(w_j \mid j \in F_\gamma) = \mathbb{P}(\mathbf{w}_\gamma)$$

whose co-ordinates are labelled by  $F_\gamma = \{j \mid \gamma^* x_j = x_j\}$ .

In the present situation, we can also consider a more general object than  $H_{\text{CR}}^{*,*}([U/S\mathbb{G}_m]; \mathbb{C})$ . For any group of finite-order diagonal symmetries  $K$  we can consider

$$H_{\text{orb}}^{*,*}([U/\mathbb{G}_m], S, K; \mathbb{C}) := \bigoplus_{\gamma \in S\mathbb{G}_m} H^{p-a(\gamma), q-a(\gamma)}(V(f_\gamma)/\mathbb{G}_m; \mathbb{C})^K,$$

where  $a(\gamma)$  is the age of the representation of  $\gamma \in S\mathbb{G}_m$  on the tangent bundle of  $[U/\mathbb{G}_m]$ . This is a natural object to consider; it arises naturally in several mirror symmetry contests and appears already in the literature: for instance, in the Gromov–Witten theory and Fan–Jarvis–Ruan theory of maximally invariant (Aut( $W$ )-invariant) cohomology classes. We refer to Theorem 28 for a statement involving  $H_{\text{orb}}^{*,*}$ .

*Remark 2.* We remark that the bi-grading of an element is, by definition, the sum of the (possibly non-integer) term  $(a(\gamma), a(\gamma))$  and the integer bi-grading of a cohomology classes in the projective variety  $\{f_\gamma = 0\}/\mathbb{G}_m$ .

*Remark 3.* This bi-grading of Chen–Ruan and orbifold cohomology is integer as soon as  $S \in \text{SL}(n; \mathbb{C})$ .

*Remark 4.* Notice that  $\gamma$  is an  $(n-1)$ -dimensional representation because  $\gamma$  acts on the fibre of the tangent bundle  $T$  of the hypersurface of  $\mathbb{P}(\mathbf{w})$  defined by  $f$ . We remark that the action of  $\gamma$  extends to the  $n$ -dimensional fibre of the tangent bundle of  $\mathbb{P}(\mathbf{w})$ . The two finite order representations are related by the following formula

$$a(\gamma: T \rightarrow T) = a(\gamma: T_{\mathbb{P}(\mathbf{w})} \rightarrow T_{\mathbb{P}(\mathbf{w})}) - a(\gamma: N \rightarrow N),$$

where  $N$  is the normal bundle of  $\mathfrak{X}$  within  $\mathbb{P}(\mathbf{w})$ . If we assume that the defining polynomial  $f$  has degree  $d$ , then  $\gamma = (h_1 \lambda^{w_1}, \dots, h_n \lambda^{w_n}) \in S\mathbb{G}_m$  acts as  $\lambda^d$  on the normal line. The reader may refer to [13, Lem. 22] for an explicit treatment in this special case.

### 3. SEMI-CALABI-YAU VARIETIES

**3.1. The defining equation.** We consider the weighted projective hypersurface defined by the vanishing locus of

$$(4) \quad W(x_1, \dots, x_n) = \sum_{i=1}^n \prod_{j=1}^n x_j^{m_{i,j}},$$

a quasi-homogeneous polynomial of weights  $w_1, \dots, w_n$  and degree  $d$ . We assume that the matrix  $M = (m_{i,j})$  admits an inverse  $M^{-1} = (m^{i,j})$ , which uniquely determines  $q_i = \frac{w_i}{d} = \sum_j m^{i,j}$ . We assume *non-degeneracy* for  $W$ ; *i.e.* regarded as a complex valued function, it satisfies  $\partial_{x_j} W(\mathbf{x}) = 0 \ \forall j$  only at  $\mathbf{x} = \mathbf{0}$ . The above conditions characterise a so-called *invertible* polynomial.

In view of mirror symmetry, the crucial extra-condition imposed to  $W$  is the  $1/2$ -Calabi-Yau condition

$$(5) \quad 2 \sum_j w_j = d,$$

or, equivalently, that the sum of the entries of  $M^{-1}$  is  $\frac{1}{2}$ .

*Remark 5.* Notice that the data  $(w_1, \dots, w_n; d)$  are uniquely determined as soon as we reduce these indices so that  $\gcd(\mathbf{w}) = 1$  (note that  $\gcd(\mathbf{w}) = \gcd(\mathbf{w}, d)$  by the  $1/2$ -CY condition).

**3.2. The Calabi-Yau  $\Sigma_W$ .** From the point of view of mirror symmetry, the defining property for a Calabi-Yau is the condition that the canonical bundle  $\omega$  is trivial. Abusing the standard terminology, we will say that a Deligne-Mumford stack is Calabi-Yau if  $\omega$  is trivial.

Clearly, the hypersurface defined by  $W$  is a smooth sub-stack of  $\mathbb{P}(\mathbf{w})$  but it is not of Calabi-Yau type. There is an obvious remedy which is to consider the two-folded cover of  $\mathbb{P}(\mathbf{w})$

$$\Sigma_W := \{x_0^2 + W\} \subset \mathbb{P}(\frac{d}{2}, w_1, \dots, w_n)$$

which has trivial canonical bundle. It is well known that this Calabi-Yau  $\Sigma_W$  admits a mirror via the standard Berglund-Hübsch construction.

**3.3. The involution  $\sigma$ .** In general, it is important to study  $\Sigma_W$  along with the involution

$$\begin{aligned} \sigma: \Sigma_W &\rightarrow \Sigma_W \\ (x_0, x_1, \dots, x_n) &\mapsto (-x_0, x_1, \dots, x_n) \end{aligned}$$

and the fixed locus of  $\sigma$ . Notice, that a point  $\mathbf{x}$  is fixed by  $\sigma$  if  $\sigma(\mathbf{x})$  equals  $\mathbf{x}$  up to the  $\mathbb{G}_m$ -action of weights  $(\frac{d}{2}, w_1, \dots, w_n)$  that define the ambient space of  $\Sigma_W$  (this is another way to say that automorphisms of a stack should be considered up to natural transformations). We provide a non-trivial example.

*Example 6.* Consider the degree 18 polynomial  $W = x_1^4 x_3 + x_3^7 x_1 + x_2^6$  whose variables have weights 4, 3, and 2. In this case, the stack  $\Sigma_W = \{x_0^2 + x_1^4 x_3 + x_3^7 x_1 + x_2^6 = 0\} \subset \mathbb{P}(9, 4, 3, 2)$  has trivial canonical bundle and the action by  $\sigma$  clearly fixes the curve  $\{x_1^4 x_3 + x_3^7 x_1 + x_2^6 = 0\}$  within the linear sub-space  $\{x_0 = 0\} = \mathbb{P}(4, 3, 2) \subset \mathbb{P}(9, 4, 3, 2)$ . It is crucial, however, to notice that  $\sigma$  fixes also  $\{x_2 = 0\}$ ; indeed if we compose  $\sigma$  with with the weighted  $(9, 4, 3, 2)$ -action of  $\lambda = -1$  we get a diagonal action fixing every variable except  $x_2$ , whose sign is changed. As a result, the fixed locus is in general larger than  $\{W = 0\} \subset \mathbb{P}(\mathbf{w})$ . In this example one can show that it is connected but not irreducible; as we shall illustrate further in Example 9

the fixed locus is not even smooth: the curve  $\{x_0 = 0, x_1^4 x_3 + x_3^7 x_1 + x_2^6 = 0\}$  and the curve  $\{x_2 = 0, x_0^2 + x_1^4 x_3 + x_3^7 x_1 = 0\}$  intersect at 5 points.

**3.4. The  $1/2$ -Calabi–Yau  $X_W$ .** There is a very natural, smooth, geometric object overlying the branch locus. We consider

$$(6) \quad X_W := \bigsqcup_{\lambda \in \mathbb{G}_m} \{(x_0^2 + W)_{\sigma\lambda} = 0\} \hookrightarrow \bigsqcup_{\lambda \in \mathbb{G}_m} \mathbb{P}(\tfrac{d}{2}, w_1, \dots, w_n)_{\sigma\lambda},$$

where  $\mathbb{P}(\tfrac{d}{2}, w_1, \dots, w_n)_{\sigma\lambda}$  is the linear sub-space whose homogeneous co-ordinates are fixed by  $\sigma\lambda$  and  $\{(x_0^2 + W)_{\sigma\lambda} = 0\}$  is its hypersurface defined by such  $\sigma\lambda$ -fixed co-ordinates.

The stack-theoretic quotient by  $\sigma$  and the image to  $\mathbb{P}(\tfrac{d}{2}, w_1, \dots, w_n)$  via the morphism forgetting  $\lambda \in \mathbb{G}_m$  yield the branch locus of the stack-theoretic quotient  $\Sigma_W \rightarrow [\Sigma_W/\sigma]$ . The stack  $X_W$  is the new geometric object defined by  $W$ : we refer to it as a  $1/2$ -Calabi–Yau because of the condition (5) and we provide a cohomological mirror to it (Theorem 17). Notice that the pullback of  $T_{[\Sigma_W/\sigma]}$  yields a coherent locally free sheaf on  $[X_W/\sigma]$ . In this way, just as for the inertia stack, each point coupled with  $\lambda \in \mathbb{G}_m$  is attached to a finite order representation of the fibre of  $T_{[\Sigma_W/\sigma]}$ , *i.e.* the action of  $\lambda \in \mathbb{G}_m$ . Since the age  $a(\sigma\lambda)$  is locally constant, it depends only on the connected component and ultimately only on  $\lambda$  (see for instance [3]). We consider the cohomology groups

$$(7) \quad H_{\sigma}^{p,q}(X_W; \mathbb{C}) := \bigoplus_{\lambda \in \mathbb{G}_m} H^{p-a(\sigma\lambda), q-a(\sigma\lambda)}(\{(x_0^2 + W)_{\sigma\lambda} = 0\}; \mathbb{C})(\tfrac{1}{2}),$$

where  $(\tfrac{1}{2})$  denotes an overall shift  $(p, q) \mapsto (p - \tfrac{1}{2}, q - \tfrac{1}{2})$ . We also consider the  $\sigma$ -invariant and the  $\sigma$ -anti-invariant parts

$$(8) \quad H_{\sigma}^{p,q}(X_W; \mathbb{C})_+ \text{ and } H_{\sigma}^{p,q}(X_W; \mathbb{C})_- \subseteq H_{\sigma}^{p,q}(X_W; \mathbb{C}).$$

**Proposition 7.** *The bi-gradings of  $H_{\sigma}^{*,*}(X_W; \mathbb{C})$ ,  $H_{\sigma}^{*,*}(X_W; \mathbb{C})_+$  and  $H_{\sigma}^{*,*}(X_W; \mathbb{C})_-$  take values in  $\mathbb{Z} \times \mathbb{Z}$  and we have*

$$H_{\sigma}^{p,q}(X_W; \mathbb{C})_+ = H_{\text{CR}}^{p+\frac{1}{2}, q+\frac{1}{2}}([\Sigma_W/\sigma]; \mathbb{C}).$$

*Proof.* The coarse space of the stack  $X_W$  is a disjoint union of projective varieties with finite quotient singularities. The grading of the Hodge decomposition of such cohomology groups takes values in  $\mathbb{Z} \times \mathbb{Z}$ . The age shifts  $a(\sigma\lambda)$  lie in  $\tfrac{1}{2} + \mathbb{Z}$  because  $\Sigma_W$  is Calabi–Yau, hence  $a(\lambda)$  is in  $\mathbb{Z}$ , and  $\sigma$  is anti-symplectic. Finally, notice that the  $\sigma$ -invariant parts of each summand of (7) is precisely the contribution to the Chen–Ruan cohomology of  $[\Sigma_W/\sigma]$  attached to group elements of the form  $\sigma\lambda$ . As argued above, these are the only group elements whose age is not integer and lies in  $\tfrac{1}{2} + \mathbb{Z}$ .  $\square$

Theorem 17 relates, via the usual mirror symmetry right-angle rotation, the above cohomology groups to the same cohomology groups of another analogous  $1/2$ -CY. This happens via a straightforward generalisation of Berglund–Hübsch construction. Before finishing this section, we would like to illustrate by some examples the above setup.

**3.5. Some examples.** The following examples illustrate the above construction in two opposite situations. In Example 8,  $X_W$  the hypersurface  $\{W = 0\}$  within  $\mathbb{P}(\mathbf{w})$ , therefore, we can set a relation between  $H_{\sigma}^{*,*}$ -cohomology and the standard cohomology of  $\{W = 0\}$ . Example 9 is the continuation of Example 6, we see that  $X_W$  can be much larger stack than the hypersurface  $\{W = 0\}$  within  $\mathbb{P}(\mathbf{w})$ ; indeed, the latter is always contained as a distinguished top-dimensional component. In this example it is interesting to point out that the





Hodge numbers in view of Example 20 which will show the mirror Hodge diamonds. The  $\sigma$ -invariant part has the following Hodge diamonds:

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & 0 & \boxed{1} & 0 \\
 1 & 0 & \boxed{4} & 0 & 0 & \boxed{10} & 1 & \boxed{10} & 0 & 0 & \boxed{35} & 0 & \boxed{232} & 0 & \boxed{35} & 0 \\
 & & 1 & & & 0 & \boxed{1} & 0 & & 0 & \boxed{0} & 1 & \boxed{0} & 0 & \boxed{35} & 0 \\
 & & & & & & 1 & & & & 0 & & 1 & & & 1
 \end{array}$$

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & 0 & \boxed{1} & 0 \\
 & & 0 & \boxed{0} & 1 & \boxed{0} & 0 \\
 0 & 0 & \boxed{126} & 0 & \boxed{2826} & 1 & \boxed{2826} & 0 & \boxed{126} & 0 & , \\
 & 0 & \boxed{0} & 0 & \boxed{0} & 1 & \boxed{0} & 0 & \boxed{0} & 0 \\
 & & 0 & \boxed{0} & 1 & \boxed{0} & 0 \\
 & & & 0 & \boxed{1} & 1
 \end{array}$$

while the  $\sigma$ -anti-invariant part is given by

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & 0 & \boxed{0} & 0 \\
 1 & 1 & \boxed{0} & 1 & 1 & \boxed{0} & 19 & \boxed{0} & 1 & 1 & \boxed{0} & 149 & \boxed{0} & 149 & \boxed{0} & 1 \\
 & & 0 & & & 0 & \boxed{0} & 0 & & 0 & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} & 0 \\
 & & & & & & 0 & & & & 0 & & 0 & & & 0
 \end{array}$$

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & 0 & \boxed{0} & 0 \\
 & & 0 & \boxed{0} & 0 & \boxed{0} & 0 \\
 1 & 0 & \boxed{0} & 976 & \boxed{0} & 3951 & \boxed{0} & 976 & \boxed{0} & 1 & . \\
 & 0 & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} & 0 \\
 & & 0 & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} & 0 \\
 & & & 0 & \boxed{0} & 0
 \end{array}$$

*Example 9.* We continue the study of the case  $W = x_1^4 x_3 + x_3^7 x_1 + x_2^6$  of degree 18 and weights 4, 3, and 2. There are four values for which the hypersurface  $\{(x_0^2 + W)_{\sigma_\lambda} = 0\}$  is nonempty. These are the fourth roots of unity.

For  $\lambda = 1$ , we examine the hypersurface defined by the restriction of  $x_0^2 + W$  to the linear sub-space defined by  $x_1, x_2$ , and  $x_3$ . This is the curve  $\{x_0 = 0, x_1^4 x_3 + x_3^7 x_1 + x_2^6 = 0\}$  fixed by  $\sigma$ . A closer study, using the standard genus formulæ within weighted projective spaces or the computation of primitive cohomology via the Milnor ring, shows that this curve has genus 3. The contribution to  $h_{\sigma}^{*,*}$  is precisely 1 in bi-degrees  $(0, 0)$  and  $(1, 1)$  and 3 in bi-degrees

$(1, 0)$  and  $(0, 1)$  (note that the age of  $T_{[\Sigma_W/\sigma]}$  is  $\frac{1}{2}$  and that this can be ignored because of the overall shift  $(\frac{1}{2})$  in the definition of  $H_\sigma^{*,*}$ ).

For  $\lambda = -1$ , we examine the hypersurface  $\{(x_0^2 + W)_{\sigma\lambda} = 0\}$  modulo  $\sigma$  defined by the restriction of  $x_0^2 + W$  to the linear subspace fixed by  $\sigma\lambda$  which acts by multiplication by  $1, 1, -1$ , and  $1$  on  $x_0, x_1, x_2$ , and  $x_3$ . This is the curve  $\{x_2 = 0, x_0^2 + x_1^4 x_3 + x_3^7 x_1 = 0\}$ ; whose coarse space is a (rational) double cover of  $\mathbb{P}(4, 2)$ . The contribution to cohomology is  $1$  in both bi-degrees  $(0, 0)$  and  $(1, 1)$ .

For  $\lambda = i$ , we notice that  $\sigma\lambda$  acts as  $\frac{1}{4}(3, 0, 3, 2)$  and that  $x_0^2 + W$  vanishes identically on the fixed space. The age of  $T_{[\Sigma_W/\sigma]}$  is  $2 - \frac{1}{2} = \frac{3}{2}$  by a straightforward application of Remark 4. This is the age of the vector bundle tangent to  $[\mathbb{P}(\frac{d}{2}, w_1, \dots, w_n)/\sigma]$  minus the age of the line bundle normal to  $[\Sigma_W/\sigma]$ . The latter is linearized by a character of weight  $\deg(x_0^2 + W) = 18$  (the reader may find a detailed analysis in [13, §5, Lem. 22]). The contribution to  $h_\sigma^{*,*}$  is  $1$  in bi-degree  $(1, 1)$ .

The analysis of the case  $\lambda = -i$  is completely analogous,  $\sigma\lambda$  acts as  $\frac{1}{4}(1, 0, 1, 2)$  and that  $x_0^2 + W$  vanishes identically on the fixed space. The age is  $1 - \frac{1}{2} = \frac{1}{2}$  (by the same argument as above). The contribution to  $h_\sigma^{*,*}$  is  $1$  in bi-degree  $(0, 0)$ .

We represent the Hodge diamond of  $H_{\text{CR}}^{*,*}(\Sigma_W; \mathbb{C})$ , which is the usual K3 surface Hodge diamond, and — within it, after the usual  $\frac{1}{2}$ -shift on both bi-degrees — that of  $H_\sigma^{*,*}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & \boxed{3} & 0 & & \\ 1 & \boxed{3} & 20 & \boxed{3} & 1 & . & \\ & 0 & \boxed{3} & 0 & & & \\ & & & 1 & & & \end{array}$$

The mirror construction we are just about to state will show why this Hodge diamond is stable with respect to a right-angle rotation: this Calabi–Yau  $\Sigma_W$ , and the  $1/2$ -Calabi–Yau  $X_W$  are self-mirrors. We illustrate this in Example 21. We record again the  $\sigma$ -invariant and anti-invariant part, in view of Example 9; we get

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & \boxed{3} & 0 & & \\ 0 & \boxed{3} & 10 & \boxed{3} & 0 & & \\ & 0 & \boxed{3} & 0 & & & \\ & & & 1 & & & \end{array}$$

and

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & \boxed{0} & 0 & & \\ 1 & \boxed{0} & 10 & \boxed{0} & 1 & . & \\ & 0 & \boxed{0} & 0 & & & \\ & & & 0 & & & \end{array}$$

*Remark 10.* In [1], the authors resolve the coarse space of  $\Sigma_W$  and study the fixed locus the involution induced by  $\sigma$  on the resolution  $Z_W \rightarrow \Sigma_W$ . The fixed locus consists of 3 connected components, smooth curves of genus 3, 0 and 0 whose Hodge diamond matches

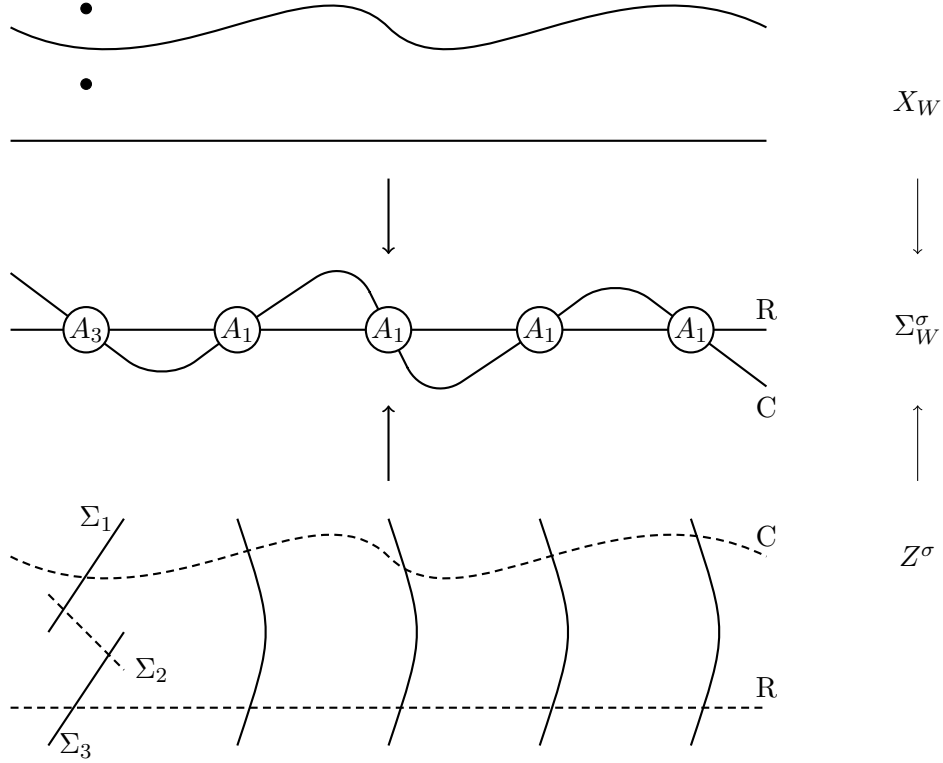


FIGURE 1. The fixed locus  $\Sigma_W^\sigma$ , its resolution  $Z^\sigma$  and the  $1/2$ -Calabi variety  $X_W$  defined by  $W = x_1^4 x_3 + x_3^7 x_1 + x_2^6$ .

that of  $H_\sigma^{*,*}(X_W; \mathbb{C})$ :

$$(9) \quad \begin{array}{ccc} & \boxed{3} & \\ \boxed{3} & & \boxed{3} \\ & \boxed{3} & \end{array} .$$

We relate  $X_W$  to this picture. Let us consider the image via the forgetful map  $p$ . This locus, fully studied in [1, Exa. 5.3], is the union of the two curves of genus 3 and 0: we have  $C = \{x_0 = 0, x_1^4 x_3 + x_3^7 x_1 + x_2^6 = 0\}$  and the rational curve  $R = \{x_2 = 0, x_0^2 + x_1^4 x_3 + x_3^7 x_1 = 0\}$  meeting at four  $A_1$ -singularities and at a single  $A_3$ -singularity of co-ordinates  $(0 : 1 : 0 : 0)$ . Notice that we have exactly  $m + 1$  points lying in the fibre of  $p : X_W \rightarrow \Sigma_W/\sigma$  over an  $A_m$ -singularity. The resolutions of these simple singularities yields chains of curves of the same length as their singularity index. It is now easy to detect the fixed locus by knowing that  $C$  and  $R$  are fixed and the chains contain alternatively  $\sigma$ -fixed subcurves and moving subcurves, where  $\sigma$  is given by  $\sigma : \mathbb{P}_z^1 \rightarrow \mathbb{P}_z^1; z \mapsto -z$ . These moving rational curves are those which share a point with  $C$  or  $R$ . Only the chain over the  $A_3$ -singularity yields a new fixed component (see Figure 1). The fixed locus is  $C \sqcup R \sqcup \Sigma_2$  and its Hodge diamond is the diamond (9) given above.

**3.6. Group actions: definition of  $\Sigma_{W,G}$  and  $X_{W,G}$ .** The automorphisms of  $\{W = 0\}$  which we consider in this paper are all acting diagonally; we write

$$\text{Aut}_W^{\text{diag}} = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{G}_m \mid W(\alpha_1 x_1, \dots, \alpha_n x_n) = W(\mathbf{x}) \ \forall \mathbf{x}\}$$

or simply  $\text{Aut}_W$  omitting the upper index “diag” as we almost always do.

The group  $\text{Aut}_W$  is finite, an immediate consequence of  $M$  being invertible. We will therefore use the notation  $\frac{1}{D}(p_1, \dots, p_n)$  for its elements. The role of the *grading element* of  $W$

$$J_W := \frac{1}{d}(w_1, \dots, w_n) \in \text{Aut}_W$$

is special. We also set

$$\text{SL}_W := \text{Aut}_W \cap \text{SL}(n; \mathbb{C}).$$

The  $1/2$ -CY condition implies that  $J_W \notin \text{SL}_W$  and  $J_W^2 \in \text{SL}_W$ .

We consider group actions  $G \subseteq \text{SL}_W$  containing  $J_W^2$

$$J_W^2 \in G \subseteq \text{SL}_W,$$

which may be regarded as defining an action on  $\mathbb{P}(\frac{d}{2}, w_1, \dots, w_n)$  which preserves the form  $dx_0 \wedge dx_1 \wedge \dots \wedge dx_n$ . Since this weighted projective space is also defined via an action (of  $\mathbb{G}_m$ ) let us specify that we refer to the action of

$$G\mathbb{G}_m = \{\lambda^{\frac{d}{2}}, \alpha_1 \lambda^{w_1}, \dots, \alpha_n \lambda^{w_n} \mid (\alpha_j)_{j=1}^n \in G, \lambda \in \mathbb{G}_m\}$$

on  $\mathbb{C}^{n+1} \setminus \mathbf{0}$ . Notice that  $G\mathbb{G}_m$  preserves the hypersurface defined by  $x_0^2 + W = 0$ . More explicitly, we look at the quotient stack

$$\Sigma_{W,G} := [\{x_0^2 + W = 0\}_{\mathbb{C}^{n+1} \setminus \mathbf{0}} / G\mathbb{G}_m]$$

which, by the condition  $G \subseteq \text{SL}_W$  will be referred to as a Calabi–Yau orbifold. This stack coincides with the quotient of  $\Sigma_W$  from the previous section modulo the faithful action of the group  $G\mathbb{C}^\times / \mathbb{C}^\times$  which can be easily shown to coincide with  $G/(J_W^2)$ . We refer to Romagny [33] and more precisely to his Remark 2.4 for a clear treatment of actions on stacks.

As before, we consider the involution  $\sigma = \frac{1}{2}(1, 0, 0, \dots, 0)$  of  $\Sigma_{W,G}$  and we analyse the  $\sigma$ -fixed locus by defining

$$X_{W,G} := \bigsqcup_{\gamma \in \sigma G\mathbb{G}_m} X_{W,G}^\gamma,$$

with

$$X_{W,G}^\gamma = \left[ \left\{ \mathbf{x} \in \mathbb{C}^{n_\gamma} \setminus \mathbf{0} \mid (x_0^2 + W(x_1, \dots, x_n))_\gamma = 0 \right\} / G\mathbb{G}_m \right]$$

where  $\mathbb{C}^{n_\gamma}$  is the vector space spanned by the co-ordinates  $x_j$  which are fixed by  $\gamma$  (i.e.  $\gamma^* x_j = x_j$ ). Again  $(x_0^2 + W)_\gamma$  denotes the restriction of the polynomial  $x_0^2 + W$  to  $\mathbb{C}^{n_\gamma}$ . Clearly each term of the union is the quotient of  $[(\mathbb{C}^{n_\gamma} \setminus \mathbf{0}) / \mathbb{G}_m]$  by the finite group  $G/(J_W^2)$ .

The quotient stack modding out the involution  $\sigma$  and the image via the morphism forgetting  $\lambda \in \mathbb{G}_m$  yields the branch locus of the stack-theoretic quotient

$$\Sigma_{W,G} \longrightarrow [\Sigma_{W,G} / \sigma].$$

We refer to  $X_{W,G}$  as a  $1/2$ -Calabi–Yau because, just as it happens for  $\Sigma_W$ , we will provide a geometric mirror for it. Notice that the pullback of  $T_{[\Sigma_{W,G}/\sigma]}$  yields a coherent locally free sheaf on  $[X_{W,G}/\sigma]$  and — by construction — a connected component labelled by  $\gamma \in \sigma G\mathbb{G}_m$  corresponds to a finite order representation of the fibre of  $T_{[\Sigma_{W,G}/\sigma]}$  at each point. The age  $a(\gamma)$  only depends on the connected component and ultimately on  $\gamma$ . We consider the cohomology group

$$(10) \quad H_\sigma^{p,q}(X_{W,G}; \mathbb{C}) := \bigoplus_{\gamma \in \sigma G\mathbb{G}_m} H^{p-a(\gamma), q-a(\gamma)}(X_{W,G}^\gamma; \mathbb{C})(\tfrac{1}{2}),$$

where  $(\frac{1}{2})$  denotes an overall shift  $(p, q) \mapsto (p - \frac{1}{2}, q - \frac{1}{2})$ . Exactly as before, we have the following result on the gradings.

**Proposition 11.** *The bi-gradings of  $H_{\sigma}^{*,*}(X_{W,G}; \mathbb{C})$ , of the  $\sigma$ -invariant  $H_{\sigma}^{*,*}(X_{W,G}; \mathbb{C})_+$  and of the  $\sigma$ -anti-invariant  $H_{\sigma}^{*,*}(X_{W,G}; \mathbb{C})_-$  all take values in  $\mathbb{Z} \times \mathbb{Z}$ . Furthermore, we have*

$$H_{\sigma}^{p,q}(X_{W,G}; \mathbb{C})_+ = H_{\text{CR}}^{p+\frac{1}{2}, q+\frac{1}{2}}([\Sigma_{W,G}/\sigma]; \mathbb{C}).$$

□

#### 4. MIRRORS OF $1/2$ -CALABI–YAU

In its standard version Berglund–Hübsch mirror symmetry produces a dual polynomial and a dual group of symmetry  $G$  out of a polynomial  $W$  defined by an invertible matrix and a group  $G$  satisfying the standard BH duality setup  $J_W \in G \subseteq \text{SL}_W$ .

Here, since  $J_W$  is not in  $\text{SL}_W$ , the group  $G$  cannot satisfy these two conditions. However, the correct generalization amounts essentially to intersect the standard Berglund–Hübsch duality with  $\text{SL}(n; \mathbb{C})$ . We get  $W^*$  and  $G^*$  satisfying the following mirror symmetry relations:

- (i) the usual right-angle rotation relation between  $X_{W,G}$  and  $X_{W^*,G^*}$  at the level of the  $\sigma$ -invariant part  $(H_{\sigma}^{*,*})_+$  and the  $\sigma$ -anti-invariant part  $(H_{\sigma}^{*,*})_-$ ;
- (ii) a right-angle rotation relation relating  $\Sigma_{W,G}$  and  $\Sigma_{W^*,G^*}$  involving on one side the bi-graded  $\sigma$ -invariant part  $H_{\text{CR}}^{*,*}(\Sigma_{W,G}; \mathbb{C})_+ \subseteq H_{\text{CR}}^{*,*}(\Sigma_{W,G}; \mathbb{C})$  and on the other side the  $\sigma$ -anti-invariant part  $H_{\text{CR}}^{*,*}(\Sigma_{W^*,G^*}; \mathbb{C})_- \subseteq H_{\text{CR}}^{*,*}(\Sigma_{W^*,G^*}; \mathbb{C})$ .

This duality is clearly compatible with the standard mirror duality between  $\Sigma_{W,G}$  and  $\Sigma_{W^*,G^*}$ , but it is richer and, as a consequence, it can also be applied as follows. The identities (i–ii) above are combined to prove that two mirror pairs of  $1/2$ -Calabi–Yau  $(W_i, G_i), (W_i^*, G_i^*)$ ,  $i = 1, 2$ , yield, via products and diagonal quotients, an ordinary mirror pair of Calabi–Yau

$$\begin{aligned} & [\Sigma_{W_1, G_1} \times \Sigma_{W_2, G_2} / (\sigma_1, \sigma_2)] \\ & \quad \updownarrow \\ & [\Sigma_{W_1^*, G_1^*} \times \Sigma_{W_2^*, G_2^*} / (\sigma_1^*, \sigma_2^*)]. \end{aligned}$$

In this section, we provide a brief, explicit, self-contained treatment of the generalised Berglund–Hübsch duality exchanging  $1/2$ -CYs, we state the mirror duality for  $1/2$ -CYs in Theorem 17 (proven in §5.3), and we prove that the standard mirror duality for the diagonal quotients above follows, see Corollary 22.

**4.1. Berglund–Hübsch for  $1/2$ -Calabi–Yau.** Let  $W$  be the polynomial attached to the matrix  $M = (m_{i,j})$  of  $1/2$ -CY type in the sense of (5). Let  $G$  be a group of determinant-1 symmetries satisfying

$$(11) \quad J_W^2 \in G \subseteq \text{SL}_W.$$

Set

$$(12) \quad W^*(x_1, \dots, x_n) := \sum_i \prod_j x_j^{m_{j,i}},$$

the polynomial attached to the matrix  $M^T = (m_{j,i})$ . It is of  $1/2$ -CY type because the sum of the weights  $w_i^*$  divided by the degree  $d^*$  is the sum of the entries of  $(M^T)^{-1}$ , i.e. the sum of the entries of  $(M)^{-1}$ , which is  $\frac{1}{2}$  since  $W$  is  $1/2$ -CY.

*Remark 12.* We recall that the Cartier duality  $\widehat{H} = \text{Hom}(H; \mathbb{G}_m)$  transforms the group  $\text{Aut}_W$  into  $\text{Aut}_{W^*}$  via a canonical isomorphism ([6, 24])

$$\widehat{\text{Aut}_W} = \text{Aut}_{W^*}.$$

Then, to each group of diagonal symmetries of  $W$  of determinant 1

$$i: G \hookrightarrow \text{SL}_W,$$

we attach a group of diagonal symmetries of  $W^*$  of determinant 1

$$(13) \quad G^* := \ker \left( \widehat{i[J_W]}: \text{Aut}_{W^*} \rightarrow \widehat{G[J_W]} \right).$$

*Remark 13.* Here,  $\widehat{i[J_W]}$  is the Cartier dual of the inclusion of  $G[J_W]$  into  $\text{Aut}_W$  and, via Remark 12, can be regarded as a homomorphism of  $\text{Aut}_{W^*}$  into  $\widehat{G[J_W]}$ .

**Proposition 14.** *If the condition  $J_W^2 \in G \subseteq \text{SL}_W$  is satisfied, then we have  $J_{W^*}^2 \in G^* \subseteq \text{SL}_{W^*}$ . Furthermore  $(G^*)^* = G$ ,  $(J_W^2)^* = \text{SL}_{W^*}$ ,  $(J_{W^*}^2) = \text{SL}_W^*$ , and if two groups  $H_1, H_2$  containing  $J_W^2$  and included in  $\text{SL}_W$  satisfy  $H_1 \subset H_2$ , then  $H_2^* \subseteq H_1^*$ .*

*Proof.* Under the canonical identification of Remark 12, imposing a character of  $\text{Aut}_{W^*}$  to vanish on  $J_W$  is equivalent to imposing the condition  $\det = 1$  within the group of diagonal symmetries of  $W^*$ . Therefore, we have

$$\ker \widehat{i[J_W]} = \ker(\widehat{i}) \cap \text{SL}_{W^*},$$

which immediately shows that this modification of the Berglund–Hübsch duality assigns to a group  $G$  a dual group in  $\text{SL}_{W^*}$ .

The condition  $J_W^2 \in G^*$  is satisfied because  $\ker(\widehat{i})$  contains  $J_W$  and  $\ker(\widehat{i}) \cap \text{SL}_{W^*}$ , has index two in  $\ker(\widehat{i})$  as a consequence of the fact that  $\ker(\widehat{i[J_W]})$  has index two in  $\ker(\widehat{i})$ , which is the same as saying  $|G[J_W]|/|G| = 2$ , which is another way to say that  $J_W^2$  lies in  $G$ .

The rest of the statement is obvious.  $\square$

*Remark 15.* We point out that the duality  $\star$  matches the standard Berglund–Hübsch duality (i.e.  $G^\vee = \ker(\widehat{i})$ ) when applied in the standard situation  $J_W \in G \subseteq \text{SL}_W$ . Indeed, in these cases the intersection with  $\text{SL}_{W^*}$  can be ignored because  $\ker(\widehat{i})$  lies in  $\text{SL}_{W^*}$ .

When we consider the automorphism group  $\text{Aut}(x_0^2 + W)$  within  $\text{GL}(n; \mathbb{C})$  we can extend each diagonal morphism and each group of diagonal automorphisms  $G \subseteq \text{Aut}(W)$  to  $\text{Aut}(x_0^2 + W)$  in an obvious way, fixing the coordinate  $x_0$ . This is what we mean in the following statement, when we regard  $G, G^* \subseteq \text{SL}_W$  as subgroups of  $\text{Aut}(x_0^2 + W)$  and  $\text{Aut}(x_0^2 + W^*)$  and we write  $J_W \in \text{Aut}(x_0^2 + W)$  and  $J_{W^*} \in \text{Aut}(x_0^2 + W^*)$ .

**Proposition 16.** *In the above 1/2-Calabi–Yau setup (5) and (11), the inclusion-reversion operation  $G \rightsquigarrow \ker(\widehat{i}_G)$  exchanges the following two diagrams*

$$\begin{array}{ccc} & G & \\ i_1 \swarrow & \downarrow i_2 & \searrow i_3 \\ G[\sigma] & G[\sigma J_W] & G[J_W] \\ i_4 \searrow & \downarrow i_5 & \swarrow i_6 \\ & G[\sigma, J_W] & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} & G^*[\sigma^*, J_{W^*}] & & & \\ i_1^* \swarrow & \uparrow i_2^* & \nwarrow i_3^* & & \\ G^*[J_{W^*}] & G^*[\sigma J_{W^*}] & G^*[\sigma^*] & & \\ i_4^* \swarrow & \uparrow i_5^* & \nwarrow i_6^* & & \\ & G^* & & & \end{array},$$

where all the arrows are injective homomorphisms.

*Proof.* Indeed,  $\sigma J_W$  and  $\sigma^* J_{W^*}$  are the grading elements of  $x_0^2 + W$  and  $x_0^2 + W^*$ . Therefore  $G[\sigma J_W]$  is dual to  $G^*[\sigma^* J_{W^*}]$ . This explains the bottom line and the upper line of the above transformations, whereas the inclusions are reversed due to Proposition 14. Finally,  $G[\sigma]$  is a direct product of  $(\sigma)$  and  $G$  (automorphism groups of summands involving disjoint sets of variables). Therefore, Berglund–Hübsch duality yields the direct product of the dual of  $\sigma$  within  $\text{Aut}(x_0^2)$ , which is trivial, with the direct product of the standard Berglund–Hübsch dual of  $G$ , which is  $G^*[J_{W^*}]$ .  $\square$

**4.2. Mirror symmetry of  $1/2$ -Calabi–Yau models.** This is the main theorem of the paper; we state it and defer the proof to the next section.

**Theorem 17.** *Let  $(W, G)$  where  $W$  satisfies (5) and  $G$  satisfies  $J_W^2 \in G \subseteq \text{SL}_W$ . Then consider the dual pair  $(W^*, G^*)$  and the respective geometric objects  $\Sigma_{W,G}, X_{W,G}, \Sigma_{W^*,G^*}, X_{W^*,G^*}$ . In all degrees  $p$  and  $q$  we have*

- (i)  $H_{\sigma}^{p,q}(X_{W,G}; \mathbb{C})_+ \cong H_{\sigma}^{n-2-p,q}(X_{W^*,G^*}; \mathbb{C})_+ \quad ;$
- (ii)  $H_{\sigma}^{p,q}(X_{W,G}; \mathbb{C})_- \cong H_{\sigma}^{n-2-p,q}(X_{W^*,G^*}; \mathbb{C})_- \quad ;$
- (iii)  $H_{\text{CR}}^{p,q}(\Sigma_{W,G}; \mathbb{C})_+ \cong H_{\text{CR}}^{n-1-p,q}(\Sigma_{W^*,G^*}; \mathbb{C})_- \quad ;$
- (iv)  $H_{\text{CR}}^{p,q}(\Sigma_{W,G}; \mathbb{C})_- \cong H_{\text{CR}}^{n-1-p,q}(\Sigma_{W^*,G^*}; \mathbb{C})_+ \quad .$

*Remark 18.* All identities on all sides involve  $\mathbb{Z} \times \mathbb{Z}$ -graded Hodge decompositions; therefore  $p$  and  $q$  may be assumed to be integers.

*Remark 19.* The above theorem will be proven in Section 5.3. The proof exploits an explicit bi-grading preserving isomorphism proven in [13] which allows to rephrase the above statements in term of a formal definition of the Landau–Ginzburg model. There, mirror symmetry holds via an explicit isomorphism introduced by Krawitz. Therefore, the above statement may be phrased by replacing “we have” by “there exist explicit isomorphisms...”.

*Example 20.* The cases examined in Example 8 fit in the above setup with  $G = (J_W^2)$ . We can verify that for  $n = 1, 2, 3, 4, 5$  the polynomial  $W^* = W = x_1^{2n} + \dots + x_n^{2n}$  paired with  $G^* = \text{SL}_W$  yields the following Hodge diamonds for  $H^{p,q}(\Sigma_{W,\text{SL}_W}; \mathbb{C})$ , where we have included  $H_{\sigma}^{p,q}(X_{W,\text{SL}_W}; \mathbb{C})$  with a shift of  $(1/2, 1/2)$ :

$$\begin{array}{ccccccc}
 & & & & & & 1 \\
 & & & & & 0 & \boxed{35} & 0 \\
 & & & & 0 & \boxed{0} & 149 & \boxed{0} & 0 \\
 2 & 1 & \boxed{4} & 1 & 1 & \boxed{1} & 20 & \boxed{1} & 1 & 1 & \boxed{1} & 1 & 1 & \boxed{1} & 1 & 1 & \boxed{1} & 1 \\
 & & & & 0 & \boxed{10} & 0 & & & 0 & \boxed{0} & 149 & \boxed{0} & 0 \\
 & & & & & 1 & & & & 0 & \boxed{35} & 0 \\
 & & & & & & & & & & 1
 \end{array}$$



$$\begin{array}{cccccccccccccccc}
& & & & & & 1 & & & & & & & & & & & \\
& & & & & & 0 & & \boxed{126} & & 0 & & & & & & & \\
& & & & & 0 & \boxed{0} & & 976 & & \boxed{0} & & 0 & & & & & \\
& & 0 & & \boxed{0} & & 0 & & \boxed{2826} & & 0 & & \boxed{0} & & & & & \\
1 & & \boxed{1} & & 1 & & \boxed{1} & & 3952 & & \boxed{1} & & 1 & & \boxed{1} & & 1 & . \\
& & 0 & & \boxed{0} & & 0 & & \boxed{2826} & & 0 & & \boxed{0} & & 0 & & & \\
& & & & 0 & & \boxed{0} & & 976 & & \boxed{0} & & 0 & & & & & \\
& & & & & & 0 & & \boxed{126} & & 0 & & & & & & & \\
& & & & & & 1 & & & & & & & & & & 
\end{array}$$

In fact, it is immediate to see that  $\mathrm{SL}_W$  is isomorphic to

$$\underbrace{\mathbb{Z}/2n\mathbb{Z} \times \dots \times \mathbb{Z}/2n\mathbb{Z}}_{n-1}$$

and that a basis is given for instance by

$$\frac{1}{2n}(1, 0, \dots, 0, 2n-1), \dots, \frac{1}{2n}(0, 0, \dots, 1, 2n-1).$$

The  $\sigma$ -invariant part is now given by the following Hodge diamonds:

[illegible]

$$\begin{array}{ccccccccccc}
& & & & & 1 & & & & & \\
& & & & 0 & \boxed{126} & 0 & & & & \\
& & 0 & \boxed{0} & 976 & \boxed{0} & 0 & & & & \\
& 0 & \boxed{0} & 0 & \boxed{2826} & 0 & \boxed{0} & & & & \\
0 & \boxed{1} & 0 & \boxed{1} & 3951 & \boxed{1} & 0 & \boxed{1} & 0 & , & \\
& 0 & \boxed{0} & 0 & \boxed{2826} & 0 & \boxed{0} & 0 & & & \\
& & 0 & \boxed{0} & 976 & \boxed{0} & 0 & & & & \\
& & & 0 & \boxed{126} & 0 & & & & & \\
& & & & 1 & & & & & & 
\end{array}$$

while the  $\sigma$ -anti-invariant part is given by

[illegible]

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \boxed{0} & & & \\
& & 0 & \boxed{0} & 0 & & \\
& 0 & \boxed{0} & 0 & \boxed{0} & 0 & \\
1 & \boxed{0} & 1 & \boxed{0} & 1 & \boxed{0} & 1 \\
& 0 & \boxed{0} & 0 & \boxed{0} & 0 & \\
& & 0 & \boxed{0} & 0 & \boxed{0} & \\
& & & 0 & & & 
\end{array}$$

Comparing with Example 8, we can note that the  $(\frac{1}{2} + \mathbb{Z}, \frac{1}{2} + \mathbb{Z})$ -graded is given by the  $(\frac{1}{2} + \mathbb{Z}, \frac{1}{2} + \mathbb{Z})$ -graded part of the mirror, rotated by a right angle, while the integer invariant and anti-invariant part are exchanged and rotated.

*Example 21.* This completes the discussion of  $W = x_1^4 x_3 + x_3^7 x_1 + x_2^6$  of degree 18 and weights 4, 3, and 2 (we refer to Examples 6 and 9 for the first part). The study of  $X_W$  and  $\Sigma_W$  fits in the present setup with  $G = (J_W^2)$ . Notice, however, that the polynomial  $W^*$  is equal to  $W$  and that  $\mathrm{SL}_W$  coincides with  $(J_W^2)$ . Therefore the above theorem predicts the fact that the Hodge diamond appearing in Example 9 is stable with respect to right angle rotations.

This symmetry is the result of the fact that the  $\sigma$ -invariant part and anti-invariant part coincide up to a right-angle rotation, and of the fact that the  $(\frac{1}{2} + \mathbb{Z}, \frac{1}{2} + \mathbb{Z})$ -graded part is itself symmetric.

**4.3. Mirror pairs of Calabi–Yau.** Theorem 17 implies the following result.

**Corollary 22.** *Let  $(W_1, G_1)$  and  $(W_2, G_2)$  be pairs where  $W_1$  and  $W_2$  are polynomials in  $n_1$  and  $n_2$  variables satisfying (5) and we have  $J_{W_i}^2 \in G_i \subseteq \mathrm{SL}_{W_i}$  for  $i = 1, 2$ . We consider the mirror pairs  $(W_1^*, G_1^*)$  and  $(W_2^*, G_2^*)$  and all the geometric data, on one side  $\Sigma_{W_i, G_i}$  with its involution  $\sigma_i$ , on the other side  $\Sigma_{W_i^*, G_i^*}$ , with its involution  $\sigma_i^*$ , for  $i = 1, 2$ . Set the  $(n_1 + n_2 - 2)$ -dimensional Calabi–Yau orbifolds*

$$\Sigma_{n_1+n_2} = [\Sigma_{W_1, G_1} \times \Sigma_{W_2, G_2} / (\sigma_1, \sigma_2)]$$

and

$$\Sigma_{n_1+n_2}^* = [\Sigma_{W_1^*, G_1^*} \times \Sigma_{W_2^*, G_2^*} / (\sigma_1^*, \sigma_2^*)].$$

Then, we have

$$H_{\text{CB}}^{p,q}(\Sigma_{n_1+n_2}; \mathbb{C}) \cong H_{\text{CB}}^{n_1+n_2-2-p,q}(\Sigma_{n_1+n_2}^*; \mathbb{C}).$$

*Proof.* The stack  $\Sigma_{n,m}$  is the quotient stack  $[U/G]$  where  $U$  is the locus within  $\mathbb{C}^{n+m+2}$  where  $(x_0^1)^2 + W_1$  and  $(x_0^2)^2 + W_2$  vanish and  $G$  is the the group whose elements are of the form

$$(\lambda^{\frac{d_1}{2}}, \alpha_1^1 \lambda^{w_1^1}, \dots, \alpha_n^1 \lambda^{w_n^1}, \mu^{\frac{d_2}{2}}, \alpha_1^2 \mu^{w_1^1}, \dots, \alpha_n^2 \mu^{w_n^1})$$

or of the form

$$(-\lambda^{\frac{d_1}{2}}, \alpha_1^1 \lambda^{w_1^1}, \dots, \alpha_n^1 \lambda^{w_n^1}, -\mu^{\frac{d_2}{2}}, \alpha_1^2 \mu^{w_1^2}, \dots, \alpha_n^2 \mu^{w_n^2}),$$

where we have  $\lambda, \mu \in \mathbb{G}_m$  and  $\alpha^i \in G_i$ ,  $\mathbf{w}^i$  is the multi-weight of  $W^i$  and  $d^i$  its degree. In order to distinguish between the two type of symmetries above, we write

$$G = G_{\clubsuit} \sqcup G_{\heartsuit}.$$

Chen–Ruan’s orbifold cohomology is a direct sum over each symmetry  $\gamma$  of one of the above forms. (We can restrict to a finite number of elements essentially because the group action is proper and there exists only a finite number of symmetries fixing a co-ordinate among

$x_0^1, x_1^1, \dots, x_n^1$  and a co-ordinate among  $x_0^2, x_1^2, \dots, x_m^2$ .) For every such symmetry  $\gamma$  we write  $(\gamma_1, \gamma_2)$  separating the first  $n+1$  co-ordinates and the remaining  $m+1$  co-ordinates. Then, the contribution to Chen–Ruan’s cohomology in bi-degree  $(p, q)$  is a cohomology group

$$H^{h,k}(V((x_0^1)^2 + W_1) \times V((x_0^2)^2 + W_1)/G) \\ = H^{h,k}(((V((x_0^1)^2 + W_1)_{\gamma_1} \times V((x_0^2)^2 + W_1)_{\gamma_2}) / (G_1 \times G_2)) / (\sigma_1, \sigma_2))$$

where  $(h, k) \in \mathbb{Z} \times \mathbb{Z}$  satisfies

$$(h, k) + (a(\gamma_1) + a(\gamma_2), a(\gamma_1) + a(\gamma_2)) = (p, q).$$

Notice that  $(V((x_0^1)^2 + W_1) \times V((x_0^2)^2 + W_1)/(G_1 \times G_2))$  equals the product of two projective varieties with finite group quotient singularities

$$V_1 \times V_2 = (V((x_0^1)^2 + W_1)_{\gamma_1}/G_1) \times (V((x_0^2)^2 + W_1)_{\gamma_2}/G_2);$$

so, the  $(h, k)$ -graded cohomology decomposes as

$$\bigoplus_{\substack{h_1+h_2=h \\ k_1+k_2=k}} \left( H^{h_1, k_1}(V_1; \mathbb{C}) \otimes H^{h_2, k_2}(V_2; \mathbb{C}) \right)^{(\sigma_1, \sigma_2)},$$

where each summand equals

$$\left( H^{h_1, k_1}(V_1; \mathbb{C})_+ \otimes H^{h_2, k_2}(V_2; \mathbb{C})_+ \right) \oplus \left( H^{h_1, k_1}(V_1; \mathbb{C})_- \otimes H^{h_2, k_2}(V_2; \mathbb{C})_- \right)$$

with  $H_+$  and  $H_-$  denoting the involution-invariant and involution-anti-invariant subspaces of  $H$ .

Summing up,  $H_{\text{CR}}^{p,q}(\Sigma_{m+n}; \mathbb{C})$  equals the direct sum of

$$\bigoplus_{\substack{\bullet=\clubsuit, \heartsuit \\ \gamma=(\gamma_1, \gamma_2) \\ \gamma \in G_\bullet}} \bigoplus_{\substack{h_1+h_2=p-a(\gamma_1)-a(\gamma_2) \\ k_1+k_2=q-a(\gamma_1)-a(\gamma_2)}} \bigoplus_{\substack{H^{h_1, k_1}(V_1; \mathbb{C})_+ \otimes H^{h_2, k_2}(V_2; \mathbb{C})_+}}$$

and

$$\bigoplus_{\substack{\bullet=\clubsuit, \heartsuit \\ \gamma=(\gamma_1, \gamma_2) \\ \gamma \in G_\bullet}} \bigoplus_{\substack{h_1+h_2=p-a(\gamma_1)-a(\gamma_2) \\ k_1+k_2=q-a(\gamma_1)-a(\gamma_2)}} \bigoplus_{\substack{H^{h_1, k_1}(V_1; \mathbb{C})_- \otimes H^{h_2, k_2}(V_2; \mathbb{C})_-}}.$$

The first term can be rewritten as

$$\bigoplus_{\substack{\bullet=\clubsuit, \heartsuit \\ p_1+p_2=p \\ q_1+q_2=q}} \bigoplus_{\substack{\gamma=(\gamma_1, \gamma_2) \\ \gamma \in G_\bullet}} \bigoplus_{\substack{h_1=p_1-a(\gamma_1) \\ h_2=p_2-a(\gamma_2) \\ k_1=q_1-a(\gamma_1) \\ k_2=q_2-a(\gamma_2)}} \bigoplus_{\substack{H^{h_1, k_1}(V_1; \mathbb{C})_+ \otimes H^{h_2, k_2}(V_2; \mathbb{C})_+}},$$

where the  $\clubsuit$ -summand yields

$$\bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\text{CR}}^{p_1, q_1}(\Sigma_{W_1, G_1}; \mathbb{C})_+ \otimes H_{\text{CR}}^{p_2, q_2}(\Sigma_{W_2, G_2}; \mathbb{C})_+,$$

whereas the  $\heartsuit$ -summand yields

$$\bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\sigma}^{p_1-\frac{1}{2}, q_1-\frac{1}{2}}(X_{W_1, G_1}; \mathbb{C})_+ \otimes H_{\sigma}^{p_2+\frac{1}{2}, q_2-\frac{1}{2}}(X_{W_2, G_2}; \mathbb{C})_+.$$

Similarly, the second term yields the direct sum of

$$\bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\text{CR}}^{p_1, q_1}(\Sigma_{W_1, G_1}; \mathbb{C})_- \otimes H_{\text{CR}}^{p_2, q_2}(\Sigma_{W_2, G_2}; \mathbb{C})_-.$$

and

$$\bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\sigma}^{p_1-\frac{1}{2}, q_1-\frac{1}{2}}(X_{W_1, G_1}; \mathbb{C})_- \otimes H_{\sigma}^{p_2-\frac{1}{2}, q_2-\frac{1}{2}}(X_{W_2, G_2}; \mathbb{C})_-.$$

By Theorem 17 applied to each term of the four formulæ above we get

$$\begin{aligned} & \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\text{CR}}^{n_1-1-p_1, q_1}(\Sigma_{W_1^*, G_1^*}; \mathbb{C})_- \otimes H_{\text{CR}}^{n_2-1-p_2, q_2}(\Sigma_{W_2^*, G_2^*}; \mathbb{C})_-, \\ & \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\sigma}^{n_1-p_1-\frac{3}{2}, q_1-\frac{1}{2}}(X_{W_1^*, G_1^*}; \mathbb{C})_+ \otimes H_{\sigma}^{n_2-p_2-\frac{3}{2}, q_2-\frac{1}{2}}(X_{W_2^*, G_2^*}; \mathbb{C})_+, \\ & \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\text{CR}}^{p_1, q_1}(\Sigma_{W_1^*, G_1^*}; \mathbb{C})_- \otimes H_{\text{CR}}^{p_2, q_2}(\Sigma_{W_2^*, G_2^*}; \mathbb{C})_-, \\ & \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H_{\sigma}^{n_1-p_1-\frac{3}{2}, q_1-\frac{1}{2}}(X_{W_1^*, G_1^*}; \mathbb{C})_- \otimes H_{\sigma}^{n_2-p_2-\frac{3}{2}, q_2-\frac{1}{2}}(X_{W_2^*, G_2^*}; \mathbb{C})_-, \end{aligned}$$

which add up to  $H_{\text{CR}}^{n_1+n_2-2-p, q}(\Sigma_{n_1+n_2}^*; \mathbb{C})$ .  $\square$

*Remark 23.* Part of the above proof is just a check of Künneth formula for Chen–Ruan cohomology, which can be found in [25] in a more general setup. We provide an explicit treatment because the present situation requires a slightly more detailed treatment of invariant and anti-invariant cohomology. This, however, can possibly phrased more directly using [25].

## 5. LANDAU–GINZBURG MODELS

In terms of Landau–Ginzburg models the four statements of Theorem 17 assemble into a single one, which is part of the theory of mirror maps of Landau–Ginzburg state spaces developed by Kreuzer [28], Krawitz [27] and Borisov [11].

We review the definitions of the state spaces. Then, we state the relevant version of the Landau–Ginzburg mirror symmetry theorem and we state the first author’s Landau–Ginzburg/Calabi–Yau correspondence proved with Ruan in [13]. Finally, we prove theorem 17 by relying on these two tools: we first detail a special case of the Landau–Ginzburg mirror symmetry theorem, which — via the LG/CY correspondence — decomposes into the four statements of Theorem 17.

**5.1. State spaces.** For any degree  $d$  quasihomogeneous nondegenerate polynomial  $W$  in the variables  $x_1, \dots, x_r$  of weights  $w_1, \dots, w_r$  (regardless of any condition on the sum of the weights such as (5) or even any invertibility condition on the defining matrix), we consider the (full) state space of the Landau–Ginzburg  $W: \mathbb{C}^N \rightarrow \mathbb{C}$

$$\mathcal{H}^{*,*}(W) := \bigoplus_{g \in \text{Aut}(W)} \text{Jac}(W_g),$$

20

where each summand is given by the Jacobi ring

$$\text{Jac}(W_g) = \left[ \mathbb{C}[x_j \mid j \in F_g] / (\partial_j W_g \mid j \in F_g) \right] \bigwedge_{j \in F_g} dx_j$$

(see §2.2 for  $F_g$ ,  $W_g$ ). Within  $\mathcal{H}(W)$ , the notation  $[g, \phi]$  specifies an element in the image of  $i_g: \text{Jac}(W_g) \rightarrow \mathcal{H}(W)$

$$[g, \phi] \in \text{Jac}(W_g) \xrightarrow{i_g} \mathcal{H}(W).$$

The bi-grading  $(p, q)$  is defined for each monomial term  $\prod_{j \in F_g} x_j^{a_j} \bigwedge_{j \in F_g} dx_j$  as

$$(14) \quad (p, q) = (\#F_g - \deg + a(g), \deg + a(g)),$$

where

$$\deg \left( \prod_{j \in F_g} x_j^{a_j} \bigwedge_{j \in F_g} dx_j \right) = \frac{1}{D} \sum_j (a_j + 1) w_j.$$

In this paper, we regard  $\mathcal{H}$  as a bigraded vector space and we never use its ring structures (*e.g.* [21]). Any diagonal symmetry  $\frac{1}{D}(p_1, \dots, p_r) \in \text{Aut}(W)$  acts on the line spanned by  $\prod_{j \in F_g} x_j^{a_j} \bigwedge_{j \in F_g} dx_j$  as the character

$$\frac{1}{D} \left( \sum_j (a_j + 1) p_j \right),$$

(a  $\mu_D$ -character with notation (3)). Notice that the action of the element  $J_W$  which we have referred to as the “grading element” is actually given by its grading: the character  $\deg$ .

This space is also referred to as “unprojected” state space in the literature (see [27]). This is because it can be decomposed and projected onto several sub-spaces playing an important role in the theory of Landau–Ginzburg models. We provide a rather general tool to identify all the sub-spaces relevant in this paper. We use  $S$  and  $K$ , both subgroups of  $\text{Aut}(W)$ ;  $S$  plays the usual role of a group of symmetries of  $\mathbb{C}^r$  preserving  $W$ , whereas  $K$  plays the role of a group selecting the right  $K$ -invariant forms (it kills the non-invariant ones). For any group of symmetry  $S \subseteq \text{Aut}(W)$  let us introduce

$$\mathcal{H}_S^{*,*}(W) = \bigoplus_{g \in S} \text{Jac}(W_g).$$

Furthermore for any group  $K \subseteq \text{Aut}(W)$  we can extract the  $K$ -invariant part

$$\mathcal{H}^{*,*}(W)^K = \bigoplus_{g \in \text{Aut}(W)} \text{Jac}(W_g)^K.$$

We can clearly combine the two constructions and get  $\mathcal{H}_S^{*,*}(W)^K$ . In the special case where  $S = K$ , we recover the Fan–Jarvis–Ruan state of a Landau–Ginzburg model  $W: [\mathbb{C}^r/S] \rightarrow \mathbb{C}$ ; here we only need the following case where  $W$  is a polynomial whose degree  $d$  equals the sum of the weights: we have

$$\mathcal{H}_{\text{FJR}}^{p,q}(W, G) := \mathcal{H}_G^{p+1, q+1}(W)^G,$$

for any group of symmetries  $G$ . (The definition of FJR state space is usually stated more generally for any  $W$ , but the overall shift (1) should be replaced by the sum of the weights divided by the degree.)

**5.2. LG mirror symmetry and LG/CY correspondence.** We now consider invertible Landau–Ginzburg potential as in (4). We still avoid imposing any condition on the sum of weights and the degree such as (5).

Let  $W$  and  $W^*$  be two non-degenerate Landau–Ginzburg potentials in  $r$  variables whose exponent matrix are the invertible matrix  $M = (m_{i,j})$  and its transposed  $M^T$ . Following the notation 3, the columns of  $M^{-1} = (m^{i,j})$  are generators of  $\text{Aut}(W)$  and the lines of  $M^{-1}$  are generators of  $\text{Aut}(W^*)$ .

The following theorem is proven in various versions in [28, 27, 11]. We follow [28] and [27], there to each monomial  $x_1^{a_1} \cdots x_r^{a_r}$  we attach a diagonal symmetry as follows

$$\gamma: x_1^{a_1} \cdots x_r^{a_r} \mapsto \prod_{j \in F_g} (m^{j,1}, \dots, m^{j,n})^{a_j}$$

where  $m^{i,j}$  are the coefficients of the inverse of the exponent matrix of  $W$  (refer to notation (3)). The right hand side lies in  $\text{Aut}(W^*)$  because the lines of the inverse matrix  $M$  span  $\text{Aut}(W^*)$ . With a slight abuse of notation identifying the form  $\prod_{j \in F_g} x_j^{a_j-1} \bigwedge_{j \in F_g} x_j$  to the monomial  $\prod_{j \in F_g} x_j^{a_j}$ , we can apply  $\gamma$  to each summand of  $\mathcal{H}(W)$ :

$$\gamma: \text{Jac}(W_g) \rightarrow \text{Aut}(W).$$

*Remark 24.* In particular,  $\gamma$  provides an equivalent interpretation of the standard Berglund–Hübsch dual group  $\ker(\widehat{\iota}: \text{Aut}(W^*) \rightarrow \widehat{G})$  attached to any  $i_G: G \hookrightarrow \text{Aut}(W)$ . We have

$$\ker(\widehat{\iota}: \text{Aut}(W^*) \rightarrow \widehat{G}) = \gamma(\{G\text{-invariant monomials}\}),$$

where the right hand side is actually Krawitz’s original formalization of the standard Berglund–Hübsch duality. To the best of our knowledge, this was the first precise formalization given to Berglund–Hübsch duality.

The following theorem is Krawitz’s main result in [27]. As we mentioned above, closely related statement can be found in [28] and [11].

**Theorem 25** (Krawitz). *We have an isomorphism*

$$\Gamma: \mathcal{H}^{p,q}(W) \rightarrow \mathcal{H}^{n-p,q}(W^*).$$

*The isomorphism attaches to each element of the form*

$$[h, \phi = \prod_{j \in F_h} x_j^{a_j-1} \bigwedge_{j \in F_h} x_j]$$

*a unique element of the form  $[\gamma(\phi), T \in \text{Jac}(W_{\gamma(\phi)}) \cap \gamma^{-1}\{h\}]$ .*  $\square$

*Remark 26.* Although we do have a weight  $1/2$  appearing in  $x_0^2 + W$ , Krawitz’s proof does apply because the restriction to weights  $< 1/2$  is needed only in the chain case (and in our case,  $x_0^2$  is Fermat).

**Corollary 27.** *For any  $i: G_1 \hookrightarrow \text{Aut}(W)$ ,  $j: H_1 \hookrightarrow \text{Aut}(W)$ , consider the dual groups via the standard Berglund–Hübsch duality  $G_2 = \ker(\widehat{\iota}: \text{Aut}(W^*) \rightarrow \widehat{G_1})$  and  $H_2 = \ker(\widehat{j}: \text{Aut}(W^*) \rightarrow \widehat{H_1})$ . Then,  $\Gamma$  yields an isomorphism*

$$\Gamma: \mathcal{H}_{H_1}^{p,q}(W)^{G_1} \rightarrow \mathcal{H}_{G_2}^{n-p,q}(W^*)^{H_2}.$$

*Proof.* We only need to show that the image of  $\mathcal{H}_{H_1}(W)^{G_1}$  via  $\Gamma$  is contained in  $\mathcal{H}_{G_2}(W)^{H_2}$ . Then, the same claim holds in the opposite sense and we conclude by Theorem 25.

Given  $[h, \phi] \in \mathcal{H}_{H_1}(W)^{G_1}$  we need to prove that the image  $[\gamma(\phi), T]$  lies in  $\mathcal{H}_{G_2}(W)^{H_2}$ . First,  $\gamma(\phi)$  lies in  $G_2$ , because, by Remark 24 we have  $\gamma(\text{Jac}(W_h)) \subseteq G_2$ . Second, the form

$T$  is  $H_2$ -invariant. Indeed this amounts to proving that  $T$  is invariant with respect to any symmetry of the form  $\gamma(M)$  for any  $H_1$ -invariant monomial  $M$ . The last claim is equivalent to showing that  $\gamma(T)$  fixes any  $H_1$ -invariant monomial  $M$ ; this is the case because we have  $\gamma(T) = h$  and  $h \in H_1$ .  $\square$

We finally recall the state space isomorphism establishing the Landau–Ginzburg/Calabi–Yau correspondence. Although in this paper we only apply this theorem to invertible polynomials, we do not need any invertibility condition on the polynomial. On the other hand, we require that all groups of symmetries involved in the statement contain  $J_W$ .

**Theorem 28** (Chiodo–Ruan [13]). *Let  $W$  be any non-degenerate polynomial of weights  $w_1, \dots, w_r$  and degree  $d = w_1 + \dots + w_r$  (Calabi–Yau condition). Let  $S$  and  $J$  be two groups of diagonal symmetries containing  $J_W$ . Then, for any  $p$  and  $q$ , we have*

$$H_{\text{orb}}^{p,q}([\{W = 0\}_{\mathbb{C}^r}/\mathbb{G}_m], S, K; \mathbb{C}) \cong \mathcal{H}_S^{p,q}(W)^K,$$

and in particular we have

$$H_{\text{CR}}^{p,q}([\{W = 0\}_{\mathbb{C}^r}/\text{SG}_m]; \mathbb{C}) \cong \mathcal{H}_{\text{FJR}}^{p,q}(W, S),$$

for any  $p$  and  $q \in \mathbb{Q}$ , where the isomorphism is compatible with any finite-order diagonal symmetry acting on  $\mathbb{C}^r$  and preserving  $W$ .  $\square$

*Remark 29.* A straightforward consequence of the two theorems above is the standard Berglund–Hübsch mirror symmetry for Calabi–Yau. In the same conditions as in Theorem 25 we restrict to the case  $K = S$ . Then, for  $K' = \ker(\hat{\iota}: \text{Aut}(W^*) \rightarrow \hat{K})$ , we have

$$\begin{aligned} \mathcal{H}_{\text{FJR}}^{p,q}(W, K) &\cong \mathcal{H}_K^{p+1, q+1}(W)^K \\ &\cong H_{K'}^{n-p-1, q+1}(W^*)^{K'} = H_{K'}^{n-2-p, q}(W^*)^{K'}(1) \cong \mathcal{H}_{\text{FJR}}^{n-2-p, q}(W^*, K'), \end{aligned}$$

which, via Theorem 28 is the standard Berglund–Hübsch mirror duality for Calabi–Yau orbifolds.

**5.3. Proof of the main theorem.** Here  $W$  is an invertible polynomial satisfying the  $1/2$ -Calabi–Yau condition (5) and  $G$  contains  $J_W^2$  and lies in  $\text{SL}_W$  as in (11). We regard  $G$  as a subgroup of  $\text{Aut}(x_0^2 + W)$ ; the grading element of

$$V := x_0^2 + W$$

is  $J_V = \sigma J_W$  where  $J_W$  is regarded as an element of  $\text{Aut}(V)$  fixing  $x_0$ .

Our proof derives from applying Krawitz’s mirror symmetry to a single state space; namely, we consider

$$\mathcal{H}_{G[\sigma, J_W]}(V)^{G[J_V]},$$

which, in the course of the proof we simply denote by  $\mathcal{H}$ . First, we point out that it decomposes as

$$\mathcal{H} = [\mathcal{H}]_G \oplus [\mathcal{H}]_{\sigma G} \oplus [\mathcal{H}]_{J_W G} \oplus [\mathcal{H}]_{J_V G},$$

where we adopt the following notation

$$[\mathcal{H}]_L := \bigoplus_{g \in L} \text{Jac}(V_g)^{G[J_V]} \quad \text{for } L = G, \sigma G, J_W G, J_V G,$$

and  $[\mathcal{H}]_L^{p,q}$  for the intersection  $[\mathcal{H}]_L \cap \mathcal{H}^{p,q}$ .

**Lemma 30.** *The state space  $\mathcal{H}$  decomposes into a  $\mathbb{Z} \times \mathbb{Z}$ -graded part,*

$$(15) \quad [\mathcal{H}]_G \oplus [\mathcal{H}]_{J_V G}$$

*and a  $(\frac{1}{2} + \mathbb{Z}) \times (\frac{1}{2} + \mathbb{Z})$ -graded part*

$$(16) \quad [\mathcal{H}]_{\sigma G} \oplus [\mathcal{H}]_{J_W G}.$$

*Proof.* We notice the following property of any form in  $\text{Jac}(W_g)^{G[J_V]}$ : their degree mod  $\mathbb{Z}$  coincides with  $a(g) \bmod \mathbb{Z}$ , which equals  $\frac{1}{2} \bmod \mathbb{Z}$  if  $g$  lies in  $\sigma G$  or  $J_W G$  and is 0 mod  $\mathbb{Z}$  if  $g$  lies in  $G$  or  $J_V G$ .  $\square$

There is another presentation of the state space  $\mathcal{H} = \mathcal{H}_{G[\sigma, J_W]}(V)^{G[J_V]}$ , which is made explicit in the following lemma.

**Lemma 31.** *There is an explicit isomorphism which preserves the bi-degree*

$$\mathcal{H}_{G[\sigma, J_W]}(V)^{G[J_V]} \cong \mathcal{H}_{G[J_V]}(V)^G.$$

*Proof.* We can decompose the right-hand side in the following way:

$$\mathcal{H}_{G[J_V]}(V)^G = \bigoplus_{g \in G} \text{Jac}(V_g)^G \oplus \bigoplus_{g \in J_V G} \text{Jac}(V_g)^G,$$

where each factor further decomposes into a space of  $J_V$ -invariant forms and a space of  $J_V$ -anti-invariant forms. Notice that the two subspaces of  $J_V$ -anti-invariant forms can be identified to  $[\mathcal{H}]_{\sigma G}$  and  $[\mathcal{H}]_{J_W G}$  respectively. The identification to  $[\mathcal{H}]_{\sigma G}$  is defined for instance in the following way:

$$\left[ \sigma g, \prod_{j \in F_h} x_j^{a_j-1} \bigwedge_{j \in F_h} x_j \right] \mapsto \left[ g, \prod_{j \in F_h} x_j^{a_j-1} dx_0 \wedge \bigwedge_{j \in F_h} x_j \right].$$

One can check that it is well-defined and that it preserves the bi-degree. The identification to  $[\mathcal{H}]_{J_W G}$  is defined in an analogous way.  $\square$

The following lemma relates  $\mathcal{H}$  to the mirror space

$$\mathcal{H}^* := \mathcal{H}_{G^*[\sigma^*, J_{W^*}]}(V^*)^{G^*[J_{V^*}]}$$

determined by  $G^*$ ,  $W^*$ , and  $V^* = x_0^2 + W^*$ .

**Lemma 32.** *There is an isomorphism  $[\mathcal{H}]^{p,q} \cong [\mathcal{H}^*]^{n-p,q}$  and, more precisely,*

$$\begin{aligned} [\mathcal{H}]_G^{p,q} &\cong [\mathcal{H}^*]_{J_{V^*} G^*}^{n-p,q}, & [\mathcal{H}]_{J_V G}^{p,q} &\cong [\mathcal{H}^*]_{G^*}^{n-p,q}, \\ [\mathcal{H}]_{\sigma G}^{p,q} &\cong [\mathcal{H}^*]_{\sigma^* G^*}^{n-p,q}, & [\mathcal{H}]_{J_W G}^{p,q} &\cong [\mathcal{H}^*]_{J_{W^*} G^*}^{n-p,q}. \end{aligned}$$

*Proof.* Proposition 16 specifies the Cartier dual of each group involved. It allows us to apply explicitly the mirror duality of Corollary 27, which yields

$$\mathcal{H}_{G[\sigma, J_W]}^{p,q}(V)^{G[J_V]} \cong \mathcal{H}_{G[J_{V^*}]}^{n-p,q}(V)^{G^*}.$$

Lemma 31 allows us to identify the right hand side with  $\mathcal{H}^*$ ; in this way we finally get

$$[\mathcal{H}]^{p,q} \cong [\mathcal{H}^*]^{n-p,q},$$

as a result of Krawitz's mirror map  $\Gamma$  and the isomorphism of Lemma 31, which we will call  $\Phi$ . Since the bi-degree transformation preserves  $\mathbb{Z} \times \mathbb{Z}$ , the expression (15) lands on the analogous expression for  $W^*$  and  $G^*$ ; we notice that on this expression, the isomorphism  $\Phi$  is simply the identity. The same goes for (16) because the gradings  $(\frac{1}{2} + \mathbb{Z}) \times (\frac{1}{2} + \mathbb{Z})$  are preserved.



We can match up the summands of (15) and (16) using the following simple observations. The isomorphism is the composition of the mirror map  $\Gamma$  and of  $\Phi$ . On the term (15), the map  $\Gamma$  transforms  $\sigma$ -invariant forms into  $\sigma$ -anti-invariant forms and  $\Phi$  is the identity; we deduce that the two summands of (15) are exchanged. On the term (16),  $\Phi$  transforms  $\sigma$ -invariant forms into  $\sigma$ -anti-invariant forms; so, ultimately,  $\Phi \circ \Gamma$  transforms  $\sigma$ -invariant forms into  $\sigma$ -invariant forms; we deduce that the two summands of (15) are preserved and this completes the proof.  $\square$

*Proof of Theorem 17.* We recall that  $J_V$  is the grading element of  $V = x_0^2 + W$  and we apply the Landau–Ginzburg/Calabi–Yau correspondence to  $\mathcal{H}$ ; we get

$$H_{\text{orb}}^{p,q}([\{x_0^2 + W\}/\mathbb{G}_m], G[\sigma, J_W], G[J_V]) \cong [\mathcal{H}]^{p,q},$$

where the left hand side is, by definition, isomorphic to

$$H_{\text{CR}}^{p,q}(\Sigma_{W,G}; \mathbb{C}) \oplus H_{\sigma}^{p-\frac{1}{2}, q-\frac{1}{2}}(X_{W,G}; \mathbb{C}).$$

Hence, considering the  $\sigma$ -invariant and anti-invariant parts, we get

$$\begin{aligned} H_{\text{CR}}^{p,q}(\Sigma_{W,G}; \mathbb{C})_+ &= [\mathcal{H}]_{J_V G}^{p,q}, \\ H_{\text{CR}}^{p,q}(\Sigma_{W,G}; \mathbb{C})_- &= [\mathcal{H}]_G^{p,q}, \\ H_{\sigma}^{p-\frac{1}{2}, q-\frac{1}{2}}(X_{W,G}; \mathbb{C})_+ &= [\mathcal{H}]_{\sigma G}^{p,q}, \\ H_{\sigma}^{p-\frac{1}{2}, q-\frac{1}{2}}(X_{W,G}; \mathbb{C})_- &= [\mathcal{H}]_{J_W G}^{p,q}. \end{aligned}$$

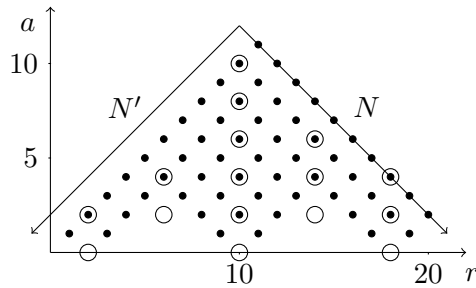
Now, Lemma 32 and the four above equations yield the claim.  $\square$

## APPENDIX

In this appendix we briefly recall the main invariants involved in Borcea–Voisin threefolds and mirror symmetry of lattice polarized K3 surfaces.

**K3 surfaces with anti-symplectic involution.** A pair  $(\Sigma, \sigma)$  formed by a K3 surface  $\Sigma$  and an anti-symplectic involution  $\sigma: \Sigma \rightarrow \Sigma$  may be regarded as a lattice polarised K3 surface; the polarisation is given by the  $\sigma$ -invariant lattice  $M = H^2(S, \mathbb{Z})^{\sigma}$  within  $H^2(S, \mathbb{Z})$ .

Nikulin [32] showed that the lattices obtained in this way are 2-elementary, their discriminant group  $M^{\vee}/M$  is isomorphic to  $(\mathbb{Z}/2)^a$  for a some  $a$ . Two-elementary lattices are classified up to isometry by three invariants: the rank of the lattice  $r$ , the rank  $a$  of  $M^{\vee}/M$  over  $\mathbb{Z}/2$ , and  $\delta \in \{0, 1\}$ , vanishing if and only if  $x^2 \in \mathbb{Z}$  for all  $x \in M^{\vee}/M$ . All the possible 79 invariants  $(r, a, \delta)$  of the lattices  $M$  arising from K3 surfaces with anti-symplectic involution are pictured here below



where a dot, resp. a circle, in position  $(r, a)$  indicates the existence of a K3 with involution whose invariants are  $(r, a, 1)$ , resp.  $(r, a, 0)$ .

Notice that the twelve cases satisfying  $r + a = 22$  or  $(r, a, \delta) = (14, 6, 0)$  are special. They are precisely the cases we need to take off for the figure to possess a symmetry with respect to the vertical axis  $r = 10$ . The explanation is mirror symmetry of lattice polarised K3 surfaces.

**Lattice mirror symmetry.** The K3 lattice  $H^2(\Sigma; \mathbb{Z})$  is isomorphic to  $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ . For any even non-degenerate lattice  $M$  of signature  $(1, \rho-1)$ ,  $1 \leq \rho \leq 19$ , admitting a primitive embedding  $M \hookrightarrow \Lambda$ , Dolgachev constructs a coarse moduli space  $\mathcal{K}_M$  of  $M$ -polarized K3 surfaces, *i.e.* pairs  $(S, j)$  where  $S$  is a K3 surface and  $j: M \hookrightarrow \text{Pic}(S)$  is a primitive lattice embedding.

The mirror symmetry of lattice polarised K3 requires preliminary condition: we assume that  $M$  has a perpendicular lattice  $M^\perp$  within  $\Lambda$  satisfying

$$(17) \quad M^\perp \cong U \oplus M'.$$

Then we refer to  $M'$  as the mirror lattice and we notice that  $(M')^\perp$  is isomorphic to  $U \oplus M$  (hence  $(M')' = M$ ). The moduli space  $\mathcal{K}_{M'}$  will be referred to as the mirror moduli space of  $\mathcal{K}_M$ .

Voisin [36] proved that the 2-elementary lattices  $M = H^2(S, \mathbb{Z})^\sigma$  which are not among the twelve special cases ( $r + a = 22$  or  $(r, a, \delta) = (14, 6, 0)$ ) mentioned above are exactly those which satisfy the preliminary condition (17). For such lattices, the mirror lattice  $M'$  has invariants  $(20 - r, a, \delta)$ . This explains the symmetry appearing within the picture given above.

Two lattice polarised K3 surfaces form a mirror pair if they are represented by two points lying in two mirror spaces  $\mathcal{K}_M, \mathcal{K}_{M'}$  for some 2-elementary lattice  $M$  with  $r + a \neq 22$  and  $(r, a, \delta) \neq (14, 6, 0)$ .

**The  $\sigma$ -fixed locus.** Because  $\sigma$  is anti-symplectic its fixed locus  $X$  is a disjoint union of  $N$  smooth curves of genera  $g_1, \dots, g_N$ . Its description is largely due to Nikulin [32]. The total genus is  $N' = \sum_{i=1}^N g_i$ . We have

$$N = \frac{r-a}{2} + 1, \quad N' = -\frac{r+a}{2} + 11,$$

except when  $(r, a, \delta) = (10, 10, 0)$  where the fixed locus is empty, *i.e.*  $N = 0$  and  $N' = 0$ . And mirror symmetry yields a symmetry along the axis  $N = N'$  interchanging  $N$  with  $N'$ . We drew the  $N$ -axis and the  $N'$ -axis in the above picture.

The invariants  $N$  and  $N'$  equal respectively  $h^{0,0}$  and  $h^{1,0}$  of  $X_\sigma$ . It turns out that the fixed locus contains at most a single component of genus  $g > 0$  except from the case  $(r, a, \delta) = (10, 8, 0)$  where the fixed locus is the union of two elliptic curves. Therefore, in all remaining cases there are  $N - 1$  rational curves and one curve of genus  $N'$  (possibly vanishing as it happens when  $r + a = 22$ ).

The Berglund–Hübsch construction identifies K3 surfaces with anti-symplectic involution. They are desingularisations of the various quotients  $\{f = 0\}/G$  with  $f(x_0, x_1, x_2, x_3) = x_0^2 + W(x_1, x_2, x_3)$  with  $W$  satisfying the  $1/2$ -Calabi–Yau condition and  $G$  a group of diagonal symmetries of determinant one. The Berglund–Hübsch mirror is the quotient  $\{f' = 0\}/G'$  induced by transposition and Cartier duality. This yields a large number of cases to study; the embedding into non-Gorenstein weighted projective spaces often complicates the resolution of singularities. Artebani, Boissière and Sarti [1] compute the corresponding invariants  $(r, a, \delta)$  in all possible cases. Out of the 79 Nikulin’s possible triples  $(r, a, \delta)$  only 29 possible triples

$(r, a, \delta)$  arise as lattice invariants of all Berglund–Hübsch mirror pairs. We notice that neither the twelve special triples without mirror, nor the single case with empty  $\sigma$ -fixed locus, nor the single case with  $\sigma$ -fixed locus given by two elliptic curves ever occur among these 29 cases. Furthermore, if the invariant attached to  $\{f = 0\}/G$  equals  $(r, a, \delta)$ , then the invariant of the Berglund–Hübsch mirror  $\{f' = 0\}/G'$  equals  $(20 - r, a, \delta)$ . This proves the compatibility of Berglund–Hübsch construction with the lattice mirror symmetry of polarized K3 surfaces.

It is easy to see that  $N = h_{\text{CR}}^{0,0}(X_{W,G}; \mathbb{C})$  and  $N' = h_{\text{CR}}^{1,0}(X_{W,G}; \mathbb{C})$  when  $n = 2$ . The fact that  $N$  and  $N'$  are exchanged, or equivalently the fact that  $(r, a)$  mirrors  $(20 - r, a)$ , can be then deduced from Theorem 17.

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